Advanced Short-Term Actuarial Mathematics Exam

Exam ASTAM

Date: Wednesday, October 25, 2023

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has 6 questions numbered 1 through 6 with a total of 60 points.

   The points for each question are indicated at the beginning of the question.

2. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions provided in this document.

3. Each question part or subpart should be answered in the provided Yellow Answer Booklet. Graders will only look at the work in the Yellow Answer Booklet.

4. The Excel file will not be uploaded for grading, and therefore will NOT BE GRADED. It should be used for looking up values for statistical functions and may be used for calculations.

5. If you use Excel for calculations, you should include as much information in the Yellow Answer Booklet as if you had used a calculator, including formulas and intermediate calculations where relevant. Written answers without sufficient support will not receive full credit.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas in the Yellow Answer Booklet. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, hand in all your written answer sheets to the Prometric Center staff. Be sure to hand in all your answer sheets because they cannot be accepted later.

© 2023 by the Society of Actuaries
475 N. Martingale Road
Schaumburg, IL 60173-2226

****BEGINNING OF EXAMINATION****
*****ADVANCED SHORT-TERM ACTUARIAL MATHEMATICS*****

1.

(*11 points*) An insurance company sells a 1-year insurance policy covering dental costs. In 2023, the policy has no deductible and no policy limit.

The number of dental visits is assumed to follow a Poisson distribution with an expected value of 1.9. The cost of each dental visit is assumed to follow a Pareto distribution with parameters of $\alpha = 4$ and $\theta = 900$. The number of dental visits and cost of each dental visit are assumed to be mutually independent.

(a) (*2 points*) Let S represent the aggregate payments in 2023 under this policy.

(i) Show that $E[S] = 570$.

(ii) Calculate the standard deviation of *S*.

In 2024, the cost of each dental visit is expected to increase uniformly by 20% due to inflation. The number of dental visits is assumed to have the same distribution in 2024 as in 2023.

(b) (*1 point*) Explain why the distribution of the cost of each dental visit in 2024 is Pareto, with parameters $\alpha = 4$ and $\theta = 1080$.

(c) (*4 points*) The insurer is considering imposing an ordinary deductible per dental visit, *d*, that would result in the insurer's expected aggregate payments being the same in 2024 as in 2023.

(i) Show that $d = 70$ to the nearest 10. You should calculate the value to the nearest 0.1.

(ii) Calculate the expected number of claim payments if the deductible in (c)(i) is applied, assuming that the number of dental visits is not affected by the deductible.

(iii) Briefly describe 2 ways in which the introduction of a deductible might impact policyholder behavior.

(d) (*2 points*) The insurer is instead considering imposing an ordinary deductible of 50 along with a maximum covered loss of *L* on each claim payment in 2024. Calculate the value of *L* that would result in the insurer's expected aggregate payments to the insurer being the same in 2024 as in 2023.

1. **Continued**

The insurer wants to estimate the aggregate loss distribution by discretizing the 2023 severity distribution, using the Method of Rounding with a span of 100.

(e) (*1 point*) Determine the probability that is assigned to a claim severity of 0.

(f) (*1 point*) Calculate $f_S(0)$.

2.

(*10 points*) An insurer is analyzing its auto insurance claims with a focus on the right tails of the distributions.

(a) (*1 point*) Losses are categorized into two categories: bodily injury and vehicle repairs.

State with reasons which of these categories is more likely to have a heavy-tailed loss distribution.

(b) (*1 point*)

(i) State whether an increasing mean excess loss function indicates a heavy-tailed or light-tailed distribution.

(ii) State whether an increasing hazard rate function indicates a heavy-tailed or light-tailed distribution.

(c) (*3 points*) You are given that $X \sim \text{Pareto}(\alpha, \theta_X)$.

(i) Prove that $e_X(d) = \dfrac{\theta_X + d}{\alpha - 1}$ for $\alpha > 1$.

(ii) Determine an expression for the hazard rate function for *X*, $h_X(x)$, in terms of α and θ_X.

(d) (*2 points*) You are given that $Y \sim \text{Weibull}(\tau, \theta_Y)$.

(i) Derive an expression for the hazard rate function for *Y*, $h_Y(y)$, in terms of τ and θ_Y.

(ii) Determine the range of values of τ for which *Y* is heavy-tailed, based on the hazard rate function.

(e) (*2 points*) You are given that $\alpha = 2$, $\tau = 1/2$, $\theta_X = 20$, and $\theta_Y = 10$. Use the limiting ratio of the survival function of *X* to the survival function of *Y* to compare their relative tail weights.

(f) (*1 point*) You are given that *Y* is in the MDA of the Gumbel distribution. Briefly describe the implication of this on the moments of *Y*.

3.

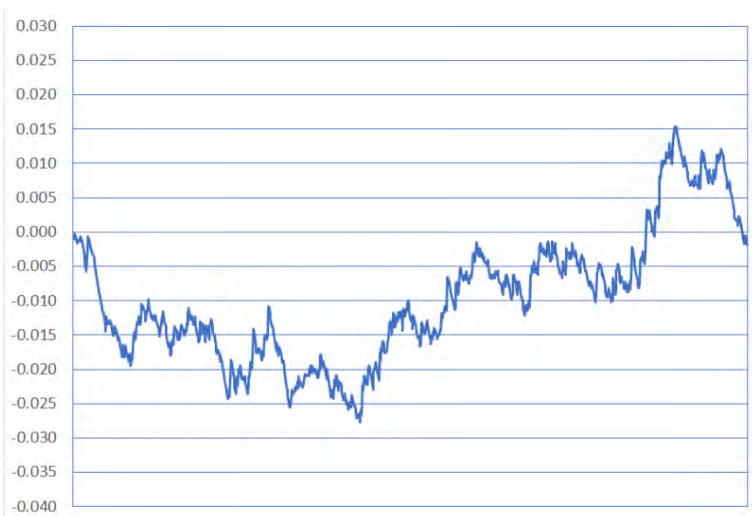
(*12 points*) An actuary is analyzing a sample of 1000 claims, as part of a review of pricing assumptions. Claims are assumed to be independent and identically distributed. Let x_i denote the amount of the i^{th} claim in the sample.

You are given that the sample mean is $\bar{x} = 1015.92$.

(a) (*3 points*) The actuary first fits an exponential distribution to the claims data using maximum likelihood estimation. You are given that the maximum likelihood estimate of the exponential distribution parameter θ is $\hat{\theta} = \bar{x}$.

(i) Show that the maximum value of the log-likelihood function under the exponential distribution is -7925 to the nearest 5. You should calculate the value to the nearest 0.1.

(ii) The actuary creates the following plot of the difference between the empirical distribution function and the fitted exponential distribution function for each data point, where the observations are sorted from low to high.



You are given that the critical value for the Kolmogorov-Smirnov test at the 5% significance level is $\dfrac{1.36}{\sqrt{n}}$.

Conduct a Kolmogorov-Smirnov goodness of fit test at the 5% significance level. You should state clearly your conclusion.

3. Continued

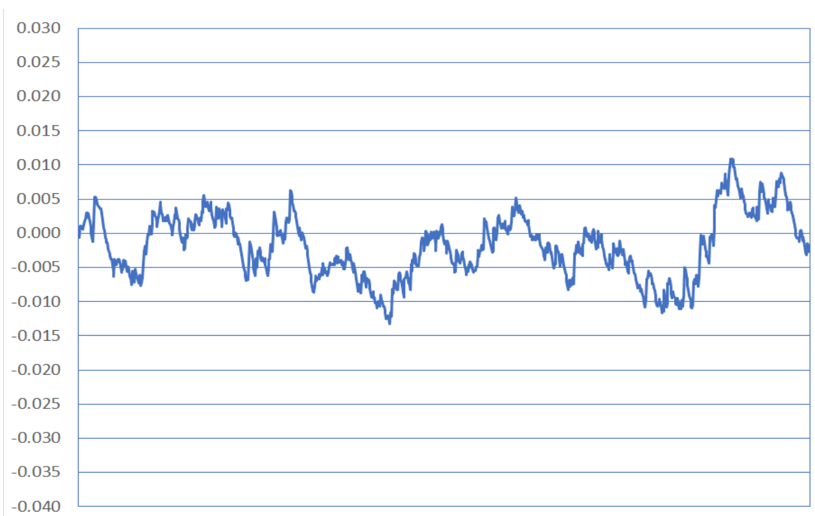
(b) (*4 points*) The actuary next fits a gamma distribution to the same claims data. You are given that:

- The maximum likelihood estimate of the α parameter is $\hat{\alpha} = 0.92463$.
- $\sum_{i=1}^{1000} \ln x_i = 6293.4$.

(i) Show that the maximum likelihood estimate of the gamma distribution parameter θ is $\hat{\theta} = 1100$ to the nearest 1. You should calculate the value to the nearest 0.01.

(ii) Show that the maximum value of the log-likelihood function under the gamma distribution is -7920 to the nearest 5. You should calculate the value to the nearest 0.1. (Hint: You can use the GAMMA function in Excel.)

(iii) The actuary creates the following plot of the difference between the empirical distribution function and the fitted gamma distribution function for each data point, where the observations are sorted from low to high.



Conduct a Kolmogorov-Smirnov goodness of fit test at the 5% significance level. You should state clearly your conclusion.

3. **Continued**

(c) (*3 points*) State with reasons which of the two distributions is preferred based on each of the following criteria:

(i) The Akaike Information Criterion (AIC).

(ii) The Schwartz Bayes Criterion (SBC/BIC).

(iii) The likelihood ratio test at the 5% significance level.

(d) (*1 point*) The actuary is concerned about the risk of claims over 6000.

(i) Calculate the probability that a randomly selected claim exceeds 6000 using the fitted exponential distribution.

(ii) Calculate the probability that a randomly selected claim exceeds 6000 using the fitted gamma distribution.

(e) (*1 point*) For the task of determining margins for solvency in reserve calculations, it is important to adequately reflect the tail risk of losses. State with reasons which of the two distributions you would recommend for this task.

4.

(*10 points*) For a portfolio of *m* independent policies, the number of claims in a year for the i^{th} policy, denoted I_i, is a Bernoulli random variable with parameter *q*.

Let n denote the total observed number of claims from the portfolio in one year so that $n = \sum_{i=1}^{m} I_i$.

(a) (*3 points*)

(i) Write down the likelihood function in terms of *q*, *n*, and *m*.

(ii) Show that the maximum likelihood estimate of *q* is the sample mean, $\hat{q} = \dfrac{n}{m}$.

An actuary decides to use a Bayesian approach to model the heterogeneity of the policies. They decide to use the Beta$(2, \beta)$ distribution, with the following density function, as the prior distribution for *q*:

$$\pi(q) = \beta(\beta+1)\, q\,(1-q)^{\beta-1} \quad \text{where} \quad 0 \le q \le 1;\ \beta > 0$$

(b) (*2 points*) Show that the posterior distribution of *q* is $\text{Beta}(n+2,\ m+\beta-n)$.

The Bayes estimate of *q* under the squared error loss function can be expressed in credibility form as $Z \times \text{sample mean } + (1-Z) \times \text{prior mean}$.

(c) (*2 points*) Show that $Z = \dfrac{m}{m+\beta+2}$.

(d) (*3 points*) The actuary uses the above Bayesian approach and distributions to analyze two portfolios, A and B, described in the following table:

Portfolio	Number of policies	Number of claims	β
A	100	20	20
B	100	10	10

(i) For each portfolio, calculate the credibility factor *Z*.

(ii) For each portfolio, calculate the Bayes estimate of *q*.

(iii) Interpret the impact of the value of β on the credibility factor.

5.

(*9 points*) An insurer has a portfolio of one-year policies with annual premiums. The claims triangle below shows the incremental claim payments for the portfolio for each of the last four accident years, by development year. Assume all incremental payments are made at the middle of each year, and claims are fully developed by the end of Development Year (DY) 3.

Accident Year (AY)	**Incremental Paid Claims by Development Year (DY)**			
	0	**1**	**2**	**3**
2019	104	38	17	5
2020	106	40	19	
2021	138	45		
2022	145			

You are given the following inflation rates, measured from mid-year to mid-year:

2019-2020	2.0%
2020-2021	4.4%
2021-2022	7.5%
2022-2023	3.0%

(a) (*1 point*) Show that the AY 2019 inflation-adjusted aggregate paid claims, expressed in terms of mid-2023 monetary units, are 190 to the nearest 10. You should calculate the value to the nearest 0.1.

(b) (*3 points*) Using the inflation-adjusted chain ladder method, show that the estimated total ultimate claims from all four accident years of the table, expressed in terms of mid-2023 monetary units, is 840 to the nearest 10. You should calculate the value to the nearest 0.1.

5. **Continued**

(c) (*3 points*) You are given that the total earned premiums were:

Year	Earned Premiums
2019	220
2020	225
2021	235
2022	275

Between 2018 and 2023 there were only two premium rate increases:

- A 4% increase on 1 July 2020
- A 6% increase on 1 October 2021

It is assumed that policies are issued uniformly and premiums are earned evenly over each year.

Show that the total earned premium for 2019-2022 at current rate levels is 1020 to the nearest 10. You should calculate the value to the nearest 0.1.

(d) (*1 point*) You are also given the following information:

(i) Total inflation-adjusted fixed expenses of 35 were incurred during 2019-2022.

(ii) Variable expenses are included in the run-off triangle figures.

(iii) The permissible loss ratio is 0.8.

Calculate the percentage increase in premium using the loss ratio method.

(e) (*1 point*) State with reasons whether the year 2019 earned premiums adjusted to current rate level would increase, decrease, or stay the same if the policies were all issued in the first three months of 2019. Assume that the unadjusted 2019 earned premiums remain at 220.

6.

(*8 points*) You are analyzing the following cumulative claims run-off triangle information using Mack's model. Assume all claims are settled by the end of Development Year (DY) 4.

	Development Year (DY) *j*				
Accident Year (AY) *i*	0	1	2	3	4
0	2296	2846	3120	3284	3336
1	2183	3001	3263	3471	
2	2201	2797	3197		
3	2226	2561			
4	2456				
$\hat{f}_j$	1.2581	?	1.0583	1.0158	
$\hat{\sigma}_j$	4.3438	?	0.4465	0.1248	
S_j	8906	8644	6383	3284	

(a) (*1 point*) Show that $\hat{f}_1 = 1.1$ to the nearest 0.1. You should calculate the value to the nearest 0.01.

(b) (*2 points*) Show that the estimated total claims cost for AY 3 is $\hat{C}_{3,4} = 3050$ to the nearest 10. You should calculate the value to the nearest 0.1.

(c) (*2 points*) Show that $\hat{\sigma}_1 = 1.6$ to the nearest 0.1. You should calculate the value to the nearest 0.01.

(d) (*3 points*)

(i) Show that the process standard deviation for outstanding claims for AY 3, using Mack's approximation, is 90 to the nearest 10. You should calculate the value to the nearest 0.1.

(ii) Calculate the square root of the estimation error for AY 3 using Mack's approximation.

(iii) Explain the difference between the variability described by the process variance and the variability described by the estimation error.

****END OF EXAMINATION****