GI ADV Model Solutions
Fall 2021

1.

Learning Objectives:

4. The candidate will understand how to apply the fundamental techniques of reinsurance pricing.

Learning Outcomes:

(4d) Apply an aggregate distribution model to a reinsurance pricing scenario.

Sources:

Basics of Reinsurance Pricing, Clark

Commentary on Question:

This question required the candidate to respond in Excel. An example of a full credit solution is in the Excel solutions spreadsheet. The solution in this file is for explanatory purposes only.

Solution:

(a) Calculate the probability that aggregate losses will be 5 billion.

Commentary on Question:

This question could be answered using recursion or first principles. Both approaches were acceptable for full credit. Both solutions are presented.

Recursion:

For negative binomial with α=1 and p=0.5, A(k) = $\sum$ [1/2 × S(i) × A(k-i)]. Let L be the loss size and A the aggregate losses. The probability (Pr) that A will be 5 billion (B) is equal to:

1/2 × [Pr (L = 1B) × Pr (A = 4B) + Pr (L = 2B) × Pr (A = 3B) +
Pr (L = 3B) × Pr (A = 2B)]
= 0.5 × [0.6 × 0.05055 + 0.3 × 0.08350 + 0.1 × 0.12000]
= 0.03369

1. Continued

First principles:

The probability of aggregate losses equal to 5 billion is the sum of the probabilities for the following events:

- 2 losses with 1 loss of 2B and 1 loss of 3B
- 3 losses with 2 losses of 1B and 1 loss of 3B
- 3 losses with 1 loss of 1B and 2 losses of 2B
- 4 losses with 3 losses of 1B and 1 loss of 2B
- 5 losses with all losses equal to 1B

$= (0.5^3 \times 2 \times 0.3 \times 0.1)$
$+ (0.5^4 \times 3 \times 0.6^2 \times 0.1)$
$+ (0.5^4 \times 3 \times 0.6 \times 0.3^2)$
$+ (0.5^5 \times 4 \times 0.6^3 \times 0.3)$
$+ (0.5^6 \times 0.6^5)$
$= 0.03369$

(b) Calculate the mean and coefficient of variation of aggregate catastrophe losses.

$E(n) = 1$
$E(L) = 0.6 \times 1 + 0.3 \times 2 + 0.1 \times 3 = 1.5$
$Var(L) = E(L^2) - E(L)^2$
$E(L)^2 = 0.6 \times 1^2 + 0.3 \times 2^2 + 0.1 \times 3^2 = 2.7$
$Var(L) = 2.7 - 1.5^2 = 0.45$

$E(A) = E(n) \times E(L) = 1.5$
$Var(A) = E(n) \times Var(L) + Var(n) \times E(L)^2 = 4.95$
Coefficient of variation of A = Std Dev(A) / E(A) = $4.95^{0.5} / 1.5 = 1.48324$

(c) Identify one disadvantage of using a recursive formula to calculate aggregate distribution probabilities.

Commentary on Question:
Only one disadvantage was required for full credit.

The calculation is inconvenient when E(n) is large.

Only a single severity distribution can be used.

2. Learning Objectives:

5. The candidate will understand methodologies for determining an underwriting profit margin.

Learning Outcomes:

(5a) Calculate an underwriting profit margin using the target total rate of return model.

(5b) Calculate an underwriting profit margin using the capital asset pricing model.

Sources:

Ratemaking: A Financial Economics Approach, D'Arcy and Dyer

Commentary on Question:

This question required the candidate to respond in Excel. An example of a full credit solution is in the Excel solutions spreadsheet. The solution in this file is for explanatory purposes only.

Solution:

(a) Calculate the target total rate of return.

Target total rate of return (TRR) = Risk-free rate + Insurer's beta × Market risk premium = 2% + 1.4 × 5% = 9.0%

(b) Calculate the underwriting profit margin.

Investable assets = IA, Owner's equity = S, Insurer's investment return = IR, Premium = P and Underwriting profit margin = UPM

TRR = (IA / S) × IR + (P / S) × UPM

UPM = (S / P) × (TRR − (IA / S) × IR)
= (500,000 / 900,000) × (9.0% − (1,200,000 / 500.000) × 6.0%)
= −3.0%

(c) Identify three differences in how Fairley's model determines *UPM* compared to the Target Total Rate of Return model.

- Fairley assumes insurers will earn the risk-free rate.
- Fairley employs the underwriting beta rather than a beta relating to company equity.
- Fairley recognizes the lag between receipt of premiums and the payment of losses and expenses.

3. Learning Objectives:

1. The candidate will understand how to use basic loss development models to estimate the standard deviation of an estimator of unpaid claims.

Learning Outcomes:

(1e) Apply a parametric model of loss development.

(1f) Estimate the standard deviation of a parametric estimator of unpaid claims.

Sources:

LDF Curve Fitting and Stochastic Reserving: A Maximum Likelihood Approach, Clark

Commentary on Question:

This question required the candidate to respond in Excel. An example of a full credit solution is in the Excel solutions spreadsheet. The solution in this file is for explanatory purposes only.

Solution:

(a) Calculate the maximum likelihood estimate of ultimate losses (*ULT*) for each of the four accident years.

For each accident year, the ultimate is the accident year total loss divided by the cumulative distribution function value at year-end 2020.

ULT2017 = 5,000 / 0.9942028 = 5,029
ULT2018 = 7,000 / 0.9747479 = 7,181
ULT2019 = 6,800 / 0.8900036 = 7,640
ULT2020 = 5,300 / 0.5208633 = 10,175

(b) Estimate the scale factor, σ^2.

For each row in the table, the expected increment (Col H) is the ULT (Col K) times [G(x) at the end of the interval (Col J) minus G(x) at the beginning of the interval (Col I)].

For each row in the table, the sigma-squared value (Col G) is (increment (Col D) minus expected increment (Col H))^2 divided by expected increment (Col H).

The scale factor, σ^2, is the sum of Col G in the table divided by five degrees of freedom which is 138.745.

3. Continued

(c) Estimate the process standard deviation of the loss reserve for all accident years combined.

The process standard deviation of the loss reserve for all accident years is the square root of (the scale factor, σ^2, times the total reserve).
Reserves for each accident year equal the ULT for each accident year minus the accident year total loss at year-end 2020.
Total Reserve =
(5,029 − 5,000) + (7,181 − 7,000) + (7,640 – 6,800) + (10,175 − 5,300) = 5,926

Process standard deviation of the loss reserve for all accident years combined = (138.745 × 5,926)^0.5 = 906.78

(d) Describe how the graph should appear if the model assumptions are satisfied.

We would expect that the residuals would be randomly scattered around the zero line for all of the ages, and that the amount of variability would be roughly constant.

(e) Determine if the model assumptions are satisfied, based on this graph.

While the variability is fairly constant from age to age, the residuals do not appear to be randomly scattered about zero, with more positive values at 12 months and more negative at 24 months.

4. Learning Objectives:

1. The candidate will understand how to use basic loss development models to estimate the standard deviation of an estimator of unpaid claims.

Learning Outcomes:

(1a) Identify the assumptions underlying the chain ladder estimation method.
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(1b) Test for the validity of these assumptions.

(1c) Identify alternative models that should be considered depending on the results of the tests.

Sources:

Measuring the Variability of Chain Ladder Reserve Estimates, Mack

Testing the Assumptions of Age-to-Age Factors, Venter

Commentary on Question:

This question required the candidate to respond in Excel. An example of a full credit solution is in the Excel solutions spreadsheet. The solution in this file is for explanatory purposes only.

Solution:

(a) State the three statistical assumptions underlying the chain ladder model.

- The conditional expected accumulated total claims amount at a given development year (DY) is the accumulated total claims amount at the previous DY times a development factor that does not vary by accident year (AY).
- The accumulated total claims amounts of different AYs are independent.
- The conditional variance of the accumulated total claims amount at a given development year is the accumulated total claims amount at the previous development year times a proportionality constant that does not vary by accident year.

(b) Demonstrate that the test statistic suggested by Mack to test for a calendar year effect is equal to 2.

Step 1: Create the triangle of age-to-age development factors.

	1-2	2-3	3-4	4-5	5-6	6-7
1	1.375	1.440	1.075	1.214	1.106	1.165
2	49.858	1.021	2.162	1.181	1.204	
3	2.039	1.654	1.085	1.223		
4	2.482	1.278	1.440			
5	4.572	1.760				
6	12.563					

4. Continued

Step 2: Create a triangle indicating if the age-to age factors in a column (each stage of development) are smaller (S) or larger (L) than the median (*) factor.

median	3.527	1.440	1.263	1.214	1.155	1.165
	j=1	j=2	j=3	j=4	j=5	j=6
j=1	S	*	S	*	S	*
j=2	L	S	L	S	L	
j=3	S	L	S	L		
j=4	S	S	L			
j=5	L	L				
j=6	L					

Step 3: Count for every diagonal (j > 1) the S's and L's, then determine the Z's (where Z is the minimum of the S_j and L_j for each j).

j	Sj	Lj	Zj
2	0	1	0
3	3	0	0
4	1	2	1
5	4	1	1
6	0	5	0

Step 4: The sum of the Z values is the test statistic.
Z = 0 + 0 + 1 + 1 + 0 = 2

(c) A test statistic equal to 2 indicates that there is a calendar year effect and implies that one of the chain ladder assumptions does not hold.

Identify that assumption.

The accumulated total claims amounts of different accident years are independent.

(d) Calculate the development terms *f*(1)-*f*(7) that minimize the sum of squared residuals.

Create the incremental loss triangle. Let AY = accident year and DY = development year.

Then for each DY d, *f*(d) is the average down the AYs (i.e., down the column).

4. Continued

Incremental loss triangle q(AY, DY)

	DY						
AY	1	2	3	4	5	6	7
1	6,012	2,257	3,638	898	2,734	1,642	2,828
2	106	5,179	111	6,270	2,116	2,817	
3	4,410	4,582	5,881	1,268	3,594		
4	4,655	6,900	3,211	6,500			
5	2,092	7,473	7,271				
6	513	5,932					
7	1,557						
d	1	2	3	4	5	6	7
f(d)	2,763.57	5,387.17	4,022.40	3,734.00	2,814.67	2,229.50	2,828.00

(e) Estimate the loss that will emerge in the next calendar year for accident years 2-7 combined.

This is the sum of the f(d) values for d = 2 to 7, which is 21,015.73

(f) Calculate the values of $f(1)$-$f(7)$ and $g(2)$-$g(7)$ that minimize the sum of squared residuals by fixing the f-values to estimate the g-values by linear regression, then fixing the g-values to estimate the next iteration of f-values by linear regression, and so on until consecutive g-values agree to two decimal places. Begin the iterative process with the f-values calculated in part (d).

The starting point for this is the set of f(d) values from part (d) representing iteration 0. The g(d) values given a set of f(d) values are calculated as follows:
$g(d) = [\sum_{i=1 \text{ to } d-1} f(i) \times q(d-i, i)] / [\sum_{i=1 \text{ to } d-1} f(i)^2]$ for d = 2 to 7 and $g(8) = 1$.

Given a set of g(d) values, the f(d) values for the next iteration are calculated as follows:
$f(d) = [\sum_{i=d \text{ to } 7} g(i+1) \times q(i-d+1,d)] / [\sum_{i=d \text{ to } 7} g(i+1)^2]$ for d = 1 to 7.

Values of f and g, with g values agreeing to two decimal places, are achieved on the thirteenth iteration:

D	1	2	3	4	5	6	7	8
f	2,878.41	7,514.33	5,738.47	5,402.98	3,207.24	2,727.52	2,828.00	
g		2.09	0.27	0.74	0.42	0.98	0.65	1.00

5. Learning Objectives:

5. The candidate will understand methodologies for determining an underwriting profit margin.

Learning Outcomes:

(5d) Allocate an underwriting profit margin (risk load) among different accounts.

Sources:

An Application of Game Theory: Property Catastrophe Risk Load, Mango

Commentary on Question:

This question required the candidate to respond in Excel. An example of a full credit solution is in the Excel solutions spreadsheet. The solution in this file is for explanatory purposes only.

Solution:

(a) Calculate the following for each of the four portfolios:

(i) Expected loss

(ii) Variance

(iii) Coefficient of variation

(i) Expected loss: Let Pr(i) be the probability for event i and L(A, i) be the loss to portfolio A for event i. Then the expected loss to portfolio A is $\sum_i \Pr(i) \times L(A, i)$.

(ii) Variance: $\sum_i \Pr(i) \times (1-\Pr(i)) \times L(A, i)^2$

(iii) Coefficient of Variation (CoV): Variance^0.5 / Expected loss

		Portfolio Q	Portfolio U	Portfolio V	Portfolio W
(i)	Expected loss	27,000	11,450	10,000	9,100
(ii)	Variance	10,090,000,000	2,860,173,750	1,284,480,000	1,097,625,000
(iii)	CoV	3.72	4.67	3.58	3.64

(b) Recommend which portfolio the reinsurance company should add if it wants to minimize the size of the total risk load. Justify your answer.

Commentary on Question:

Risk load is directly proportional to variance, so one should add the portfolio that produces the minimum variance for the combined portfolio.

5. **Continued**

For each of the combined portfolios (Q+U, Q+V and Q+W) we calculate the variance as $\sum_i \Pr(i) \times (1-\Pr(i)) \times L(U+\text{added portfolio}, i)^2$.

	Variance		
Event	Q+U	Q+V	Q+W
1	3,490,644,375	3,369,240,000	4,129,897,500
2	2,574,609,375	4,719,000,000	2,817,750,000
3	4,610,410,000	566,440,000	784,000,000
4	6,023,160,000	9,900,000,000	9,219,127,500
Total	16,698,823,750	18,554,680,000	16,950,775,000

The minimum variance is with Q and U, so company should add portfolio U.

(c) Calculate the renewal risk loads for portfolio Q and the portfolio you recommended be added in part (b).

Total risk load for Q+U = $\lambda \times$ Var(Q+U) = 0.00002 ×16,698,823,750 = 333,976.

For the Covariance Share method, the Covariance Share of Q for Q+U
= $\sum_i [L(Q, i) / (L(Q, i) + L(U, i))] \times (Var(Q+U, i) - Var(Q, i) - Var(U, i))$.

Event	Covariance Share of Q+U for Q
1	1,025,368,421
2	76,800,000
3	454,639,175
4	428,365,385
Total	1,985,172,981

The Cov. Share renewal risk load for Q = $\lambda \times$ [Var (Q) + Cov. Share for Q]
= 0.00002 × (10,090,000,000 + 1,985,172,981) = 241,503.

The Cov. Share renewal risk load for U = Total risk load for Q+U less the Cov. Share renewal risk load for Q = 333,976 – 241,503 = 92,473.

6. Learning Objectives:

2. The candidate will understand the considerations in selecting a risk margin for unpaid claims.

Learning Outcomes:

(2b) Identify the sources of uncertainty underlying an estimate of unpaid claims.

(2c) Describe methods to assess this uncertainty.

Sources:

A Framework for Assessing Risk Margins, Marshall, et al.

Commentary on Question:

This question required the candidate to respond in Excel for parts (c) and (d). An example of a full credit solution for these parts is in the Excel solutions spreadsheet. The model solutions in this file for parts (c) and (d) are for explanatory purposes only.

Solution:

(a) Identify the source of uncertainty to which each of the following belongs:

(i) uncertainty from changes to the process of setting up case reserves

(ii) insurance process too complex for any model to fully capture

(iii) unavailability of data required to conduct a credible valuation

(iv) randomness associated with the insurance process compromising the ability to select appropriate parameters

(v) uncertainty of claim costs arising from catastrophes

(vi) pure effect of the randomness associated with the insurance process

(i) III (external systemic risk)
(ii) II (internal systemic risk)
(iii) II (internal systemic risk)
(iv) I (independent risk)
(v) III (external systemic risk)
(vi) I (independent risk).

(b) Provide two reasons why stochastic modeling techniques do not enable a complete analysis of all sources of uncertainty.

6. Continued

Commentary on Question:
There are several reasons that can be provided. Only two reasons were required for full credit. The model solution is an example of a full credit solution providing two reasons.

A good stochastic model will fit the past data well and, in doing so, fit away most past systemic episodes of risk external to the valuation process, leaving behind largely random sources of uncertainty.

A stochastic model is highly unlikely to incorporate uncertainty arising from sources internal to the actuarial valuation process, i.e., internal systemic risk.

(c) Calculate the coefficient of variation for each risk source for the total insurance liabilities.

Commentary on Question:
The following abbreviations are used here: IND = independent risk, INT = internal systemic risk, EXT = external systemic risk, CoV = coefficient of variation, ρ(X) is the correlation between CL and PL for risk source X, A(CL) and A(PL) represent the amount of liabilities for CL and PL respectively, and A = A(CL) + A(PL).

For each risk source, CoV for CL and PL combined is given by:
$[\text{CoV(CL)}^2 \times (\text{A(CL)/A})^2 + \text{CoV(PL)}^2 \times (\text{A(PL)/A})^2$
$+ 2 \times \rho(X) \times \text{CoV(CL)} \times \text{CoV(PL)} \times (\text{A(CL)/A}) \times (\text{A(PL)/A})]$^0.5

	IND	INT	EXT
CoV by risk source	3.94%	3.75%	4.04%

(d) Calculate the amount of the risk margin for the total insurance liabilities at the 75% adequacy level.

Total CoV = [3.94%^2 + 3.75%^2 + 4.04%^2]^0.5 = 6.78%

Risk Margin = A × Total CoV × z-value at 75^{th} percentile of normal distribution
= 15,000 × 6.78% × 0.674 = 685.1

7. **Learning Objectives:**

3. The candidate will understand excess of loss coverages and retrospective rating.

Learning Outcomes:

(3d) Explain retrospective rating in graphical terms.

(3e) Explain Table M and Table L construction in graphical terms.

Sources:

The Mathematics of Excess of Loss Coverages and Retrospective Rating – A Graphical Approach, Lee

Commentary on Question:

This question required the candidate to respond in Excel. An example of a full credit solution is in the Excel solutions spreadsheet. The solution in this file is for explanatory purposes only.

Solution:

(a) Determine the following values:

(i) net insurance charge, I

(ii) maximum premium, G

(i) $I = [b - e + (C - 1)E\ ] / C = [15 - 30 + (1.25 - 1)100] / 1.25 = 8$

(ii) Because the minimum premium is the basic premium, the insurance savings is 0, so $\phi(r_G\ ) = I / E = 8 / 100 = 0.08$ and
$G = b + CEr_G = 15 + 1.25 \times 100 \times 1.20 = 165.$

7. Continued

(b) Fill in the missing values in Table M.

Using the relation $\psi(r) = \phi(r) + r - 1$, and the fact that the normal distribution is symmetric, yields:

Table M		
r	$\phi(r)$	$\psi(r)$
0.00	1.00	0.00
0.20	0.80	0.00
0.40	0.61	0.01
0.60	0.43	0.03
0.80	0.28	0.08
1.00	0.16	0.16
1.20	0.08	0.28
1.40	0.03	0.43
1.60	0.01	0.61
1.80	0.00	0.80
2.00	0.00	1.00

8. **Learning Objectives:**

4. The candidate will understand how to apply the fundamental techniques of reinsurance pricing.

Learning Outcomes:

(4b) Calculate the price for a property per risk excess treaty.

Sources:

Basics of Reinsurance Pricing, Clark

Commentary on Question:

This question required the candidate to respond in Excel. An example of a full credit solution is in the Excel solutions spreadsheet. The solution in this file is for explanatory purposes only.

Solution:

(a) Estimate the experience rating loss cost as a percentage of the subject premium.

For each loss, calculate the developed trended layer loss as follows:

- The untrended loss amount times the trend factor gives the trended loss. The trend factor is 1.05^(2022 – AY for loss).
- Take the layer portion of the trended loss to get the trended layer loss.
- Apply the LDF to the trended layer loss to get the developed trended layer loss. Note that the LDF for AY 2018 layer losses is 1.00, the LDF for AY 2019 layer losses is 1.25 and the LDF for AY 2020 layer losses is 1.50.

The rate is the sum of the developed trended layer losses, 4,194,542, divided by the total subject premium for the three years, 3 × 5,000,000. This equals 28.0%.

(b) Define free cover.

This refers to an experience rating in which no losses trend into the highest portion of the layer being priced.

(c) Calculate a revised loss cost as a percentage of the subject premium using these exposure factors to estimate the cost of free cover.

Select 2,000,000 excess of 1,000,000 for experience rating and 1,000,000 excess of 3,000,000 as free cover for exposure rating. Experience rating gives 28.0% for 2,000,000 excess of 1,000,000. For each layer, calculate the percentage of insured value for the top and bottom of the layer and obtain the exposure factors for these percentages using the table provided. Then calculate the difference of these exposure factors for each layer. The rate for the free cover is the experience rate times the ratio of the exposure factor difference for the free cover layer to the exposure factor difference for the experience rating layer. This equals 7.4%.
The revised loss cost is then 28.0% + 7.4% = 35.3%.

8. Continued

(d) Assess whether the loss cost percentage you calculated in part (c) would be appropriate for pricing coverage on properties with insured values of 12 million.

This would not be appropriate as the factors used are based on properties with insured values of 6 million.