Inference for Logistic-type Models for the Force of Mortality

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Presented at the Living to 100 and Beyond Symposium
Orlando, Fla.
January 7-9, 2008

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Abstract

Logistic-type models for the force of mortality like those introduced by Perks or Kannisto provide better fit to mortality data of people aged over 85 than Makeham's model where the force of mortality increases exponentially with age. However, the difficulty in estimating their parameters by the maximum likelihood method makes their use less popular among actuaries.

For Kannisto's model, we propose a weighted least-squares estimator which can easily be calculated with any regression software; the estimator is shown to be consistent, asymptotically unbiased and normally distributed. For Perks' model, using a Taylor's series expansion, the estimation problem is again reduced to a least-squares problem. The various estimators proposed in the paper are compared numerically using Canadian mortality data.

1. Introduction

In traditional actuarial models like those of Gompertz and Makeham, the force of mortality increases exponentially with age, and is an unbounded function. Perks (1932) and Kannisto (1992) have proposed logistic-type models, which better adjust to experience at ages over 85 observed recently in industrialized countries (see Thatcher, Kannisto and Vaupel, 1988). In those models, the force of mortality reaches an asymptote as the age increases to infinity. However, actuaries have not widely accepted those models for pricing annuities, one reason being the difficulty in estimating their parameters (four for Perks' and two for Kannisto's model) and also their variance-covariance matrix, by commonly used statistical methods.

In this paper, we first review existing models for the force of mortality (Section 2). We then show in Section 3, for Kannisto's model, how the logit of the force of mortality can be used to obtain a linear model, from which the two parameters can be easily estimated and their asymptotic properties derived from ordinary linear regression theory. In Section 4, we tackle the more difficult problem of estimating the four parameters of Perks' model. Using a value estimated for one of the parameters from Kannisto's model, and a limited Taylor's series for the logit of the force of mortality, the remaining three parameters could be estimated from normal linear regression.

Besides yielding estimators much easier to obtain than with the maximum likelihood method, another important advantage of this method is that a preliminary opinion over the adequacy of Kannisto's model may first be made by judging if a set of points is linear or not, before the estimation stage. If the set of points is clearly not linear, Kannisto's model is not appropriate and there is no need to estimate the parameters; contrarily to the method of maximum likelihood where the parameters must first be estimated and the fit then tested using a chi-square goodness-of-fit statistic by comparing observed values with the expected values from the model.

To reduce the variance of the estimators in Kannisto's model, weighted least-squares estimators can be calculated by taking into account the variance-covariance matrix of the errors. This will be considered in Section 5. We then apply the estimation method presented in the paper to Canadian mortality data given in Doray (2002) and compare the values obtained with those from maximum likelihood (Section 6).

2. Models for the Force of Mortality

The models presented in this section were extensively reviewed in Doray (2002). One of the first models used in actuarial science for the force of mortality μ_x at age x assumed that it was an exponential function of the attained age. Gompertz (1825) used the two-parameter function

$$\mu_x = Be^{\mu x}.$$

To take into account the force of accidental death, Makeham (1860) added an extra parameter, assumed to be independent of age, to Gompertz' model and obtained

$$\mu_x = A + Be^{\mu x}.$$

This is equivalent to assuming that if X, the lifetime of a person, has a Gompertz distribution, and Y, the time to a fatal accident, an exponential distribution, and the random variables X and Y are independent, then the minimum of X and Y has a Makeham distribution. This is an example of a shock model described in Bowers et al. (1997). Makeham's curve was used to extend mortality curves at extreme ages, and also because it possessed the the property of uniform seniority.

The British actuary Perks (1932) developed a model which did not receive as much attention in North America as the above two models. In his logistic model, the force of mortality at age x is given by the four-parameter function

$$\mu_x = \frac{A + Be^{\mu x}}{1 + Ce^{\mu x}}.$$

By assuming that the parameter $A = 0$ in the logistic model, Beard (1963) obtained the three-parameter model

$$\mu_x = \frac{Be^{\mu x}}{1 + Ce^{\mu x}}.$$

Kannisto (1992), a demographer, used the simple 2-parameter model

$$\mu_x = \frac{Be^{\mu x}}{1 + Be^{\mu x}}.$$

Those three models (logistic, Beard and Kannisto) follow a logistic-type curve for the force of mortality, i.e., as x increases, μ_x tends asymptotically to a constant. This asymptote is equal to 1 for the Kannisto's model and B/C for the Beard and logistic models. Note that the Gompertz ($A = 0,\ C = 0$), Makeham ($C = 0$), Beard ($A = 0$) and Kannisto ($A = 0,\ B = C$) models are all special cases of the logistic model, and by the principle of parsimony, should be preferred if they fit equally well as Perks' model.

Beard (1971) showed that the logistic model can arise in a heterogeneous population where each member has a Makeham force of mortality and where the parameter B varies among individuals according to a gamma distribution. This Makeham-gamma model is a frailty model. Thatcher et al. (1998) mention that the logistic model can also be considered as a shock model: if the lifetime X follows a Beard distribution, the time to an accident Y is exponentially distributed and the random variables X and Y are independent, then $\min(X, Y)$ follows a Perks distribution.

Thatcher et al. (1998) fit the Gompertz, logistic, Kannisto and Weibull models as well as the Heligman & Pollard (1980) model

$$q_x = \frac{Be^{\mu x}}{1 + Be^{\mu x}}$$

and the quadratic model

$$\ln \mu_x = a + bx + cx^2$$

to mortality data of aged people in 13 industrialized countries for the periods 1960-70, 1970-80, 1980-90 and for the cohort born in 1871-80. They used the maximum likelihood method to estimate the parameters of the models and their asymptotic variance-covariance matrix. The data used were deaths at ages 85 and over for the quadratic model and ages 80 and over for all the other models. The 13 countries included in the study were Austria, Denmark, England and Wales, Finland, France, West Germany, Iceland, Italy, Japan, the Netherlands, Norway, Sweden and Switzerland. The best fit was consistently provided by the Kannisto and logistic models for all countries in each period and for the cohort data.

All the models listed above produce very close values of μ_x at ages 80 to 95. After age 95, the Gompertz and Makeham forces of mortality continue to increase exponentially with age, while for the Kannisto, Beard and logistic models, μ_x tends asymptotically to a constant as x increases.

3. OLS Estimators for Kannisto's Model

In regresssion analysis, the data are often transformed so that the usual assumption of normality of the errors is satisfied. Popular transformations include the logarithmic, log-log, complementary log-log, probit and logit transformation (see McCullagh and Nelder, 1989).

If p is between 0 and 1, the logit of p is defined as

$$\text{logit}(p) = \ln \frac{p}{1-p}.$$

Here are some properties of the logit transform, which can easily be shown:

1) if $p \to 0$, $\text{logit}(p) \to -\infty$.

2) if $p \to 1$, logit$(p) \to +\infty$.

3) logit(1/2)= 0.

4) logit $(1-p) = \ln(1-p)/p$= -logit(p).

The logit transform of p, also simply called the logit of p is therefore a continuous function, which covers the whole real line, the same range as that of a normal distribution.

Some authors (see for example Thatcher et al. 1998) have used the fact that the logit of the force of mortality for Kannisto's model

$$\begin{aligned}\text{logit}(\mu_x) &= \ln(\tfrac{\mu_x}{1-\mu_x}) \\ &= \ln\left(\tfrac{Be^{\mu x}}{1+Be^{\mu x}} / \tfrac{1}{1+Be^{\mu x}}\right) \\ &= \ln(Be^{\mu x}) \\ &= \ln B + \mu x \qquad (1)\end{aligned}$$

is a simple function of the age x to estimate the two parameters B and μ. Note that the logit of μ_x can only be calculated if μ_x is between 0 and 1; we have seen in Section 2, that for Kannisto's model, μ_x tends asymptotically to 1, so that $\frac{\mu_x}{1-\mu_x}$ is positive and its logarithm can always be taken.

As we have seen in Doray (2002), the simple formula

$$q_x \cong 1 - e^{-\mu_{x+1/2}}$$

obtained with the midpoint rule

$$\int_x^{x+1} \mu_y dy \cong \mu_{x+1/2}$$

is often used in demography to estimate the force of mortality from a life table, with

$$\mu_{x+1/2} \cong -\ln(1-q_x) = -\ln p_x. \qquad (2)$$

This approximation gives values very close to the values obtained from the exact formula

$$\begin{aligned}p_x &= \exp\left[-\textstyle\int_x^{x+1} \mu_y dy\right] \\ &= \left(\tfrac{1+Be^{\mu x}}{1+Be^{\mu(x+1)}}\right)^{1/\mu}.\end{aligned}$$

For Kannisto's model, with the same values of the parameters as the ones obtained by Thatcher et al. (1998), the relative difference between the exact value of q_x and the approximate one obtained using the midpoint rule is only 0.03 percent for a male aged 80 and 0.0008 percent at age 100. We will therefore use the demographic assumption in the rest of this paper.

By inserting equation (2) in equation (1), we obtain

$$\begin{aligned} \text{logit}(\mu_{x+1/2}) &= \ln\left(\tfrac{-\ln p_x}{1+\ln p_x}\right) \\ &= \ln B + \mu(x+1/2) \end{aligned}$$

By defining $\alpha = \ln B$, the logit of $\mu_{x+1/2}$ is seen to be a linear function of the parameters α and μ. The probability p_x can be easily estimated by $\hat{p}_x$ from a life table. This suggests the linear model

$$\ln\left(\frac{-\ln \hat{p}_x}{1+\ln \hat{p}_x}\right) = \alpha + \mu(x+1/2) + \epsilon_x, \qquad (3)$$

where ϵ_x is a random error.

The estimator $\hat{p}_x$ has approximately a normal distribution with mean p_x if the number of persons alive at age x is large so that the errors ϵ_x have an asymptotic normal distribution with mean 0. Assuming that they have a constant variance equal to σ^2, the two parameters α and μ can be estimated from ordinary least-squares (OLS) theory.

The OLS estimators $\hat{\alpha}$ and $\hat{\mu}$ are consistent for α and μ and asymptotically are unbiased and normally distributed, so that confidence intervals can be constructed and tests of hypothesis on the value of the parameters can also be performed.

Let us assume that Kannisto's model is appropriate over the range $x = \{a, a+1, \ldots, a+b\}$, with $n = b+1-a$. Model (3) can be rewritten in vector and matrix form as

$$Y = X\theta + \epsilon,$$

where the vectors Y, θ, ϵ and the matrix X are defined as

$$Y_{n\times 1} = (Y_1, \ldots, Y_n)^{'},$$

$$\theta = (\alpha, \mu)^{'},$$

$$\epsilon_{n\times 1} = (\epsilon_1, \ldots, \epsilon_n)^{'},$$

the design matrix X

$$X_{n\times 2} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ a+0.5 & a+1.5 & \ldots & a+b+0.5 \end{pmatrix}^{'},$$

$$\text{and } Y_i = \ln\left(\frac{-\ln \hat{p}_i}{1+\ln \hat{p}_i}\right).$$

Assuming that the vector ϵ has mean 0 and the errors are independent with constant variance σ^2, from normal linear theory (see Weisberg (1985) or Montgomery and Peck (1992)), we know that the least-squares estimator $\bar{\theta}$ of θ

$$\bar{\theta} = (X'X)^{-1}X'Y$$

has a normal distribution with mean θ and variance-covariance matrix $\sigma^2(X'X)^{-1}$.

By taking into account the unequal variances of $\hat{p}_x$, and therefore those of ϵ_x, we can obtain weighted least-squares (WLS) estimators of α and μ, which will have the same properties (consistency, asymptotic unbiasedness and normality) as the OLS estimators, but with a smaller variance. This will be done in Section 5 and numerical values will be compared in Section 6.

4. Estimation for Perk's Model

As stated in Section 2, Thatcher et al. (1998) observed that the best fit to mortality data collected in 13 industrialized countries for the three decennial periods 1960-70, 1970-80, 1980-90, and for the cohort born in 1871-1880 was consistently provided by Kannisto's model and the logistic or Perks' model.

For Perks' model,

$$\mu_x = \frac{A + Be^{\mu x}}{1 + Ce^{\mu x}},$$

there are four parameters to estimate, which makes the model much more difficult to use in practice.

The function $\frac{\mu_x}{1-\mu_x}$ is equal to

$$\begin{aligned} \tfrac{\mu_x}{1-\mu_x} &= \tfrac{A+Be^{\mu x}}{1+Ce^{\mu x}} \times \tfrac{1+Ce^{\mu x}}{(1-A)+(C-B)e^{\mu x}} \\ &= \tfrac{A+Be^{\mu x}}{(1-A)+(C-B)e^{\mu x}} \\ &= \tfrac{\beta+\gamma e^{\mu x}}{1-\delta e^{\mu x}}, \end{aligned}$$

where $\beta = A/(1-A), \gamma = B/(1-A)$ and $\delta = (C-B)/(1-A)$.

Using a limited one-term Taylor's series expansion for $(1 - \delta e^{\mu x})^{-1}$, we obtain

$$\begin{aligned} \tfrac{\mu_x}{1-\mu_x} &\cong (\beta + \gamma e^{\mu x})(1 + \delta e^{\mu x}) \\ &= \beta + (\gamma + \beta\delta)e^{\mu x} + \gamma\delta e^{2\mu x} \\ &= \beta + \eta e^{\mu x} + \psi e^{2\mu x}, \end{aligned}$$

where $\eta = \gamma + \beta\delta$ and $\psi = \gamma\delta$.

If the parameter μ were known, then $\frac{\mu_x}{1-\mu_x}$ would be a linear function of the three parameters β, η and ψ.

Looking at Thatcher et al. (1998) estimates for the parameter μ (called b in his book), we see that using ages 80-98 for males, it takes values in the range 0.10 – 0.11 for Kannisto's model and around 0.10 for the logistic model. For females, μ equals around 0.12 for Kannisto's model and 0.12 – 0.13 with the logistic model.

We therefore propose to use the value estimated from Kannisto's model in Perks' model for the parameter μ. Let us denote by μ_0 this known value.

The function $\frac{\mu_x}{1-\mu_x}$ is then seen to be a linear function in all its parameters, now suggesting the linear model

$$\frac{-\ln \hat{p}_x}{1+\ln \hat{p}_x} = \beta + \eta e^{\mu_0(x+1/2)} + \psi e^{2\mu_0(x+1/2)} + \epsilon_x,$$

where the errors ϵ_x are assumed to be independent, with mean 0 and constant variance σ^2, and μ_0 is known.

Proceeding again as in Section 3, the parameters (β, η, ψ) can be estimated from ordinary least-squares. This will be investigated in Section 6.

5. WLS Estimators for Kannisto's Model

In Section 3, to calculate the OLS estimators from model (3), we assumed that the errors ϵ_x had a constant variance σ^2.

In this section, we will take into account the fact the errors ϵ_x and ϵ_y, $x \neq y$ do not have a constant variance and are not independent of each other, to obtain WLS estimators of the parameters α and μ for Kannisto's model. However, the errors do still have an asymptotic normal distribution with mean 0.

Since $\hat{p}_x$ has an asymptotic normal distribution with mean p_x and variance $p_x(1-p_x)/l_x$, we can obtain the asymptotic distribution of a function of $\hat{p}_x$ using the δ-theorem (see Lawless (1982)).

Let us define the function $h(p)$

$$h(p) = \ln\left(\frac{-\ln p}{1+\ln p}\right).$$

Calculating its derivative, we obtain

$$h'(p) = \left[p(\ln p)(1+\ln p)\right]^{-1},$$

from which, by the δ-theorem,

$$\mathrm{Var}\left[\ln\left(\frac{-\ln \hat{p}}{1+\ln \hat{p}}\right)\right] \cong (h'(p))^2|_{p=\hat{p}}\mathrm{Var}(\hat{p})$$

$$= \frac{p(1-p)/l_x}{[p(\ln p)(1+\ln p)]^2}.$$

$$\text{So } \mathrm{Var}(\epsilon_x) \cong \frac{1-p_x}{l_{x+1}[(\ln p_x)(1+\ln p_x)]^2},$$

which is estimated by

$$\frac{1-\hat{p}_x}{l_{x+1}[(\ln \hat{p}_x)(1+\ln \hat{p}_x)]^2}. \qquad (4)$$

To calculate the covariance between the errors ϵ_x and ϵ_y, where we suppose $x < y$, we use the fact that $(\hat{p}_x, \hat{p}_y)$ has a trinomial distribution with parameters (l_x, p_x, p_y), and Covariance $(\hat{p}_x, \hat{p}_y) = -p_x p_y / l_x$. By the δ-theorem again,

$$\mathrm{Cov}(\epsilon_x, \epsilon_y) \cong \left(\frac{\partial h(p_x)}{\partial p_x} \frac{\partial h(p_y)}{\partial p_y} \right) \Bigg|_{(\hat{p}_x, \hat{p}_y)} \mathrm{Cov}(\hat{p}_x, \hat{p}_y).$$

Calculating the two partial derivatives and estimating p by $\hat{p}$, $\mathrm{Cov}(\epsilon_x, \epsilon_y)$ can be estimated by

$$\frac{-1}{l_x[(\ln \hat{p}_x)(1+\ln \hat{p}_x)](\ln \hat{p}_y)(1+\ln \hat{p}_y)]}. \qquad (5)$$

The linear model of Section 3 now becomes

$$Y = X\theta + \epsilon,$$

where the vector of errors ϵ has an asymptotic normal distribution with mean 0 and variance-covariance matrix Σ, with diagonal elements given by expression (4) and off-diagonal elements given by (5).

For Kannisto's model, the weighted least-squares (WLS) estimator of $\theta = (\alpha, \mu)', \theta^* = (\alpha^*, \mu^*)'$ is equal to

$$\theta^* = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y,$$

and has an aymptotic normal distribution with mean vector θ and variance-covariance matrix $(X'\Sigma^{-1}X)^{-1}$.

The OLS and WLS estimators of θ will be compared numerically in the next section.

6. Numerical Results

To compare the various estimators possible for Kannisto's model, we will use Canadian mortality data. Table 1 contains the numbers of males and females living at ages 80 and over for the cohort born in the period 1888-1892. Those numbers, extracted from Doray (2002), were calculated with the method of extinct generations (all persons who died before age 105 have been observed). The unknown number of people born in the years 1888-1892, still alive on 1/1/1998, who would be at least 105 years old, is not counted with the method of extinct generations. However, it is believed that this number is small and that it would not affect the parameter estimates of the mortality curves too much. Table 2 contains the probabilities q_x calculated for those two cohorts.

Tables 3 and 4 contain the results of the various estimation methods, for males and females, for Kannisto's model, computed with MATHEMATICA. The estimator $\hat{\theta}$ is the MLE (the values are taken from Doray (2002), after reparametrization); $\bar{\theta}$ is the OLS estimator from Section 3, while θ^* is the WLS estimator from Section 5. We calculated another estimator $\tilde{\theta}$ by using only the diagonal elements (equation (4)) in matrix Σ, and not the covariance terms (equation (5)). There are then a lot fewer elements to calculate and Σ becomes a diagonal matrix whose inverse can be calculated much more easily.

In conclusion, for Kannisto's model, the various estimators produce values very close to each other and we therefore recommend the use of the simplest procedure, OLS. The standard errors are larger than with WLS, but this procedure requires the calculation of 190 covariances, in addition to 20 variances.

The first two columns of Table 5 contain the probabilities q_x estimated with Kannisto's model and the OLS estimator. From those probabilities, we constructed the life tables for males and females, from age 80 to 100, using as radix, the values l_{80} from Table 1. The fit is excellent for males and females,

TABLE 1 Cohort 1888-92

Age	Males	Females
80	113437	150715
81	102557	141024
82	92132	131291
83	81763	121063
84	71852	110661
85	62454	100310
86	53809	90189
87	45827	80325
88	38591	71039
89	32230	62231
90	26699	53924
91	21625	46027
92	17294	38821
93	13578	32136
94	10428	26187
95	7816	20894
96	5702	16268
97	4113	12411
98	2881	9285
99	1937	6751
100+	1311	4723

TABLE 2 Calculated Probabilities q_x

Age	Males	Females
80	0.0959	0.0643
81	0.1017	0.0690
82	0.1125	0.0779
83	0.1212	0.0859
84	0.1308	0.0935
85	0.1384	0.1009
86	0.1483	0.1094
87	0.1579	0.1156
88	0.1648	0.1240
89	0.1716	0.1335
90	0.1900	0.1464
91	0.2003	0.1566
92	0.2149	0.1722
93	0.2320	0.1851
94	0.2505	0.2021
95	0.2705	0.2214
96	0.2787	0.2371
97	0.2995	0.2519
98	0.3277	0.2729
99	0.3232	0.3004

TABLE 3 Estimated Parameter Values for Males

Estimator	α (s.e.)	μ (s.e.)
$\hat{\theta}$	-9.37522 (0.0718)	0.08922 (7.745E-4)
$\bar{\theta}$	-9.35411 (0.0114)	0.0889989 (1.264E-4)
$\tilde{\theta}$	-9.24997 (0.00145)	0.087767 (1.752E-5)
θ^*	-9.23617 (0.00127)	0.087145 (1.5673E-5)

TABLE 4 Estimated Parameter Values for Females

Estimator	α (s.e.)	μ (s.e.)
$\hat{\theta}$	-10.7428 (0.0555)	0.10053 (6.3616E-4)
$\bar{\theta}$	-10.7377 (0.0009611)	0.100516 (1.0657E-5)
$\tilde{\theta}$	-10.7361 (0.0006992)	0.100497 (8.4367E-6)
θ^*	-10.7352 (0.0005872)	0.100484 (7.2567E-6)

TABLE 5 q_x and l_x Values Estimated from Kannisto's Model

Age	q_x^M	q_x^F	l_x^M	l_x^F
80	0.0958	0.0641	113437	150715
81	0.1033	0.0701	102572	141057
82	0.1113	0.0767	91977	131164
83	0.1198	0.0838	81741	121104
84	0.1287	0.0914	71952	110958
85	0.1382	0.0996	62690	100816
86	0.1481	0.1084	54027	90773
87	0.1586	0.1178	46023	80932
88	0.1695	0.1279	38724	71396
89	0.1810	0.1385	32159	62267
90	0.1928	0.1498	26340	53641
91	0.2051	0.1618	21261	45603
92	0.2178	0.1743	16900	38226
93	0.2309	0.1875	13218	31562
94	0.2443	0.2012	10166	25645
95	0.2580	0.2154	7682	20486
96	0.2720	0.2301	5700	16073
97	0.2861	0.2453	4150	12374
98	0.3003	0.2608	2963	9339
99	0.3147	0.2766	2073	6903
100+			1421	4994

both for the values q_x and l_x.

For Perks' model, if we assume that the parameter μ takes the values (from Tables 3 and 4)

μ_0=0.09 for males and

μ_0=0.10 for females,

the matrix $(X'X)$ is badly conditioned and its inverse may contain significant numerical errors. A one-term Taylor's series expansion is therefore not appropriate and we do not report the values of the estimated parameters. In a future paper, we plan to study non-linear regression for Perks' model.

Acknowledgments

The author gratefully acknowledges the financial support of the Natural Sciences and Engineering Research Council of Canada.

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