On the Application of the Wilkie Model to the TSX Price Index

— Presentation on 42nd Actuarial Research Conference

Chao Qiu

Department of Math and Statistics
University of Calgary

August 3rd, 2007

Outline

Introduction

- The Wilkie Model is the first stochastic investment model of long term returns of multiple assets designed for actuarial application. This model was first published by Wilkie (1986).

Introduction

- The Wilkie Model is the first stochastic investment model of long term returns of multiple assets designed for actuarial application. This model was first published by Wilkie (1986).
- Many applications of Wilkie's hierarchy structural modeling method were done late on, such as Mulvey and Thorlacius (1996), Sharp (1992), and Tomson (1996).

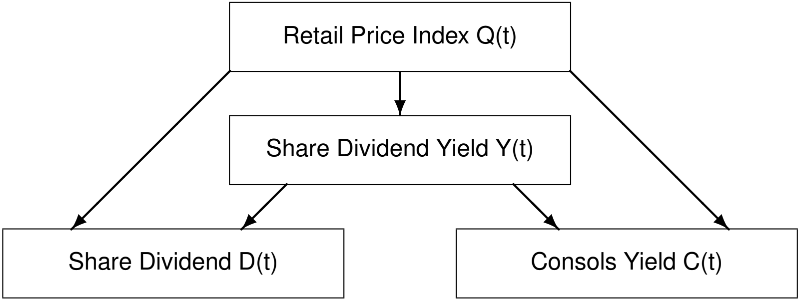
Introduction

- The Wilkie Model is the first stochastic investment model of long term returns of multiple assets designed for actuarial application. This model was first published by Wilkie (1986).
- Many applications of Wilkie's hierarchy structural modeling method were done late on, such as Mulvey and Thorlacius (1996), Sharp (1992), and Tomson (1996).
- Critics and adjustment to the Wilkie Model are also very active , such as Huber (1995), Chang and Wang (1998), and Whitten and Thomas (1999).

Outline

Cascade structure of Wilkie's Model

$$
\begin{aligned}
T_t &= D_t + \frac{D_t}{Y_t} \quad \& \quad P_t = \frac{D_t}{Y_t} \\
R_t &= \log(\frac{T_t}{P_{t-1}}), \quad \text{where } R_t \text{ is the TSX Total Return yield}
\end{aligned}
$$



Formulas of Four Components

- Model of Retail Price Index $Q(t)$:

$$\begin{aligned}\nabla \log Q(t) &= QMU + QA(\nabla \log Q(t-1) - QMU) \\ &\quad + QSD * QZ(t)\end{aligned}$$

- Model of dividend yield $Y(t)$

$$\begin{aligned}\log Y(t) &= YW * \nabla \log Q(t) + YN(t) \\ YN(t) &= \log YMU + YA(YN(t-1) - \log YMU) + YE(t)\end{aligned}$$

Formulas of Four Components

- Model of dividend index $D(t)$:

$$\begin{aligned} \nabla \log D(t) &= DW\left(\frac{DD}{1-(1-DD)B}\right)\nabla \log Q(t) + DX * \nabla \log Q(t) \\ &\quad + DMU + DY * YE(t-1) + DE(t) + DB * DE(t-1) \\ \text{where :} &\quad \frac{DD}{1-(1-DD)B} \quad \text{is exponential weighted average} \end{aligned}$$

- Model of Consols yield $C(t)$:

$$\begin{aligned} C(t) &= CW\left(\frac{CD}{1-(1-CD)B}\right)\nabla \log Q(t) + CN(t), \\ \log CN(t) &= \log CMU + CA * (\log CN(t-1) - \log CMU) \\ &\quad + CY * YE(t) + CSD * CZ(t) \end{aligned}$$

Log-likelihood of the Wilkie Model for TSX Yield

$$
\begin{aligned}
\sum_t \log f(R_t|\mathcal{F}_{t-1}) &= \sum_t \log(D_t + D_t Y_t) - \sum_t \log(D_t) \\
&\quad + \sum_t \log f(\log(Y_t)|\mathcal{F}_{t-1}) + \sum_t \log f(\nabla \log(D_t)|\mathcal{F}_{t-1})
\end{aligned}
$$

where:

$$\sum_t \log f(\log(Y_t)|\mathcal{F}_{t-1}) \quad : \quad \text{Log-likelihood of logarithmic dividend yield}$$

$$\sum_t \log f(\nabla \log(D_t)|\mathcal{F}_{t-1}) \quad : \quad \text{Log-likelihood of logarithmic dividend force}$$

Prediction with the Wilkie Model

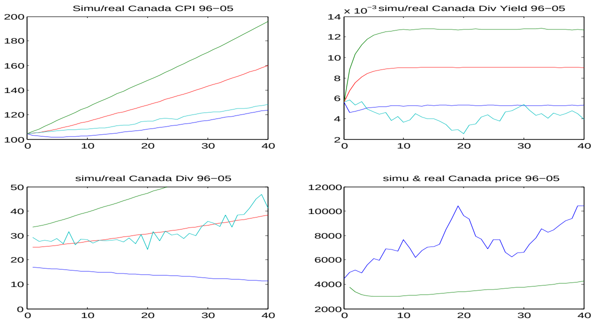


Figure: Prediction for 10 years based on Canadian data 1956–1995

ILN Model

- Independent log-normal model for short term data : $T_t = T_0 * e^{\mu t + \sigma W(t)}$, where $W(t)$ is a standard Brownian motion.

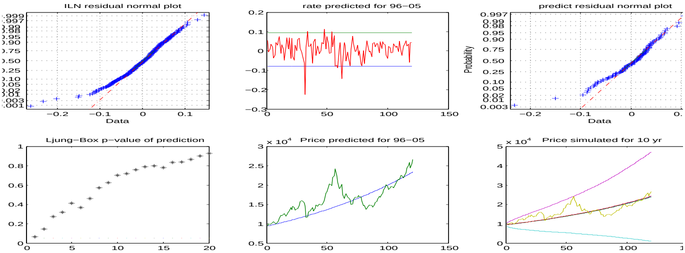


Figure: ILN model for monthly data of TSX

ARIMA–GARCH Model

- ARMA(1) model: $R_t - a(R_{t-1} - \mu) = \mu + \sigma\varepsilon_t + \theta\sigma\varepsilon_{t-1}$
  GARCH(1,1) model: $R_t = \mu + \sigma\varepsilon_t, \quad \sigma_t^2 = a_0 + a_1(R_{t-1} - \mu)^2 + \beta\sigma_{t-1}^2$.
- Fitted ARMA(1,1)+GARCH(1,1) for monthly Canadian data and ARIMA(0,0,0) x $(0,0,1)^5$ for quarterly data.

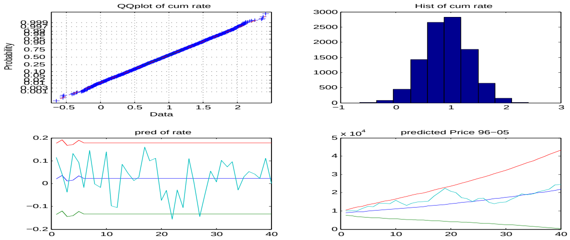


Figure: Forecast of 96–05 for quarterly Canadian data

RSLN2 Model

- Log–Normal Regime Switching (RSLN) model: $R_t|\rho_t \sim N(\mu_{\rho_t}, \sigma^2_{\rho_t})$, where ρ represents the regime, $P_{i,j}$ represents transition probabilities between regimes.

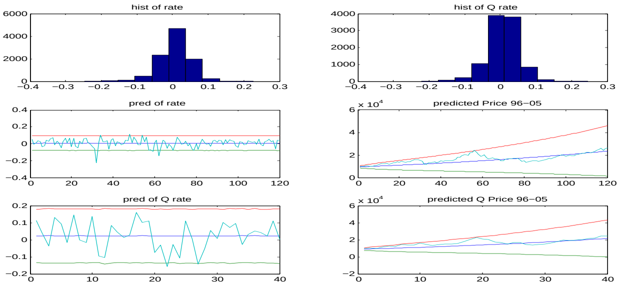
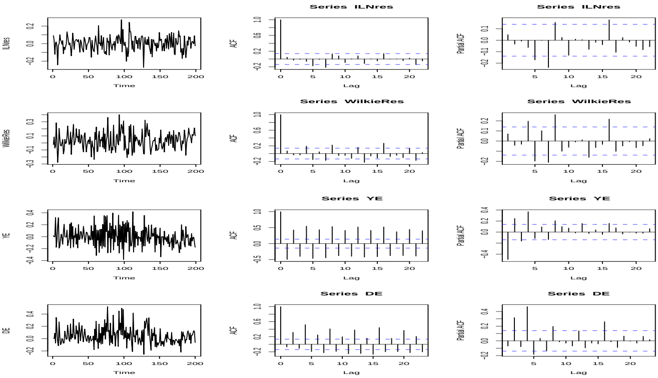


Figure: Forecast of 96–05 for 10–year Canadian data

Residual Analysis

The residuals of ARIMA-GARCH model are white noise. The following are the residuals of ILN, the Wilkie Model, and *YE* and *DE* values.



Comparison of Various Models

Table: Comparison of Fitted Models for Quarterly TSX Yield 1956-2005

Statistics	ILN	ARMA(0,0)x(0,1)^5	RSLN	the Wilkie Model
# of Parameters	2	2	6	10
logL	218.43	220.44	224.28	212.58
LRT p-value	0.0197	0.104	N/A	N/A

Comparison of Various Models

Table: Comparison of Simulation of Monthly TSX Yield over 10 Years

Statistics	ILN	ARMA(1,1)	ARMA–GARCH(1,1)	RSLN
Minimun value	-0.1986	-0.2153	-0.3558	-0.3463
2.5 percentile	-0.0806	-0.0803	-0.0817	-0.0958
5 percentile	-0.0663	-0.0661	-0.0663	-0.0677
Pr(crash)	0	0	0.0008	0.0088

- Prob(crash) is based on the event in October 1987 when the TSE index crashed with the historical low yield value of -0.2552. For monthly data, $\Pr(\text{Crash}) = \Pr\{\min_{1 \leq t \leq 120} Y_t \leq -0.2552\}$

Comparison of Various Models

Table: Comparison of Simulation of Quarterly TSX Yield over 10 Years

Statistics	ARMA(0,0)x(0,1)[5]	RSLN	the Wilkie Model
mean	0.0231	0.0229	0.0145
std.dv	0.0811	0.0807	0.2556
Kurtosis	3.0082	3.6518	3.8208
Minimun value	-0.3700	-0.4215	-0.6826
2.5 percentile	-0.1356	-0.1586	-0.3709
5 percentile	-0.1099	-0.1258	-0.3220
10 percentile	-0.0806	-0.0864	-0.2590

Outline

Calculation of Dividend Yield

- The calculation under discrete assumption of dividend yield is as follows:

$$D_t = (T'_t - T'_{t-1})\frac{P_{t-1}}{T'_{t-1}} - (P_t - P_{t-1})$$

where T'_t : TSX Total Return Index, P_t : TSX Price Index

Dividend is assumed to be discrete at the end of each period

$$Y_t = \frac{D_t}{P_t} = \frac{(T'_t - T'_{t-1})(P_{t-1})}{T'_{t-1}P_t} - \frac{P_t - P_{t-1}}{P_t}$$

- The calculation under continuous assumption is as follows:

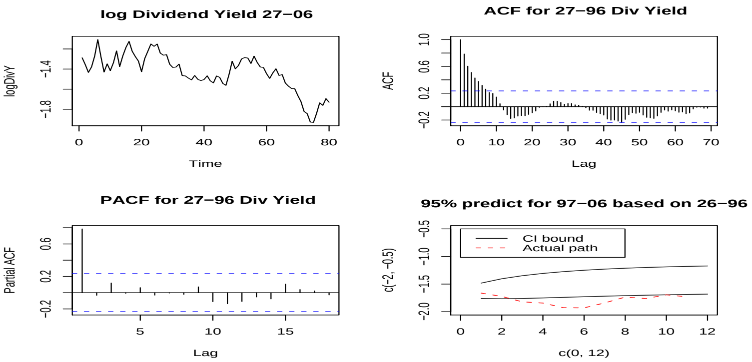
$$Y'_{t-1} = \log(\frac{T'_t}{T'_{t-1}}) - \log(\frac{P_t}{P_{t-1}})$$

$$D_t = P_{t-1} * (e^{Y'_{t-1}} - 1)$$

$$Y_t = \log(\frac{D_t}{P_t} + 1)$$

Prediction of US Market Dividend Yield

The following plots are based on the data series for US dividend yield $Y(t)$ of 1927 – 2006. The first 70 years data are used to modeling and the last 10 years data are used to compare the prediction.



High AR Coefficient for Dividend Yield Model

Table: Fitted AR Coefficients of $YN(t)$ in Dividend Yield Model for US Market

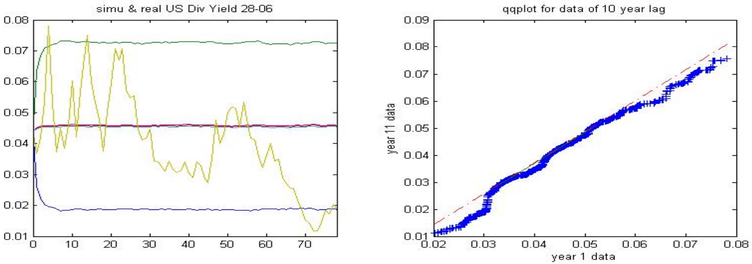
Statistcs	1926–1989	1926–2006
Quarterly Data AR1	0.77	0.93
Yearly Data AR1	0.82	0.95

Table: AR Coefficients of $YN(t)$ for Canadian Data

Statistcs	1956–1995	1956–2005
Quarterly Data AR1	0.63	0.91
Yearly Data AR1	0.85	0.97

- Comments: From the above tables, we can see that the time series models for the dividend yield is becoming not very stationary in recent years.

Decreasing Trend of Dividend Yield



The Q–Q plot of dividend yield based on monthly S&P 1926–2005 shows that the percentiles of year 11 (y coordinate) are smaller than those of year 1 (x coordinate), which is verified by paired–rank test and regression test.

Dividend for Price

- Based on Variance Bound Test:

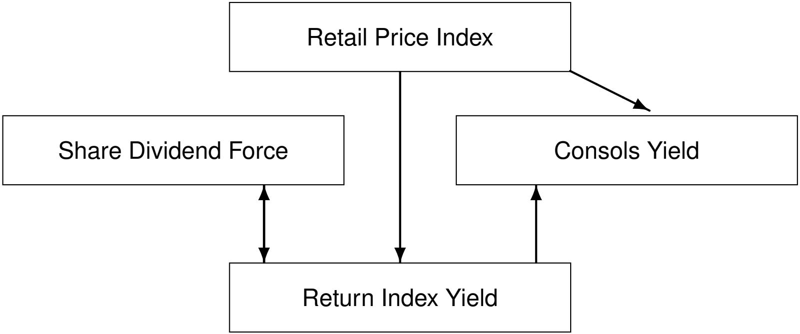
$$
\begin{aligned}
p_t &= E_t(\beta d_{t+1} + \beta^2 d_{t+2} + \ldots + \beta^{n+1} d_{t+n+1} + \beta^{n+1} p_{t+n+1}) \\
\text{let} p_t^* &= \sum_{n=1}^{\infty} \beta^n d_{t+n} \\
\text{therefore} V(p_t) &\leq V(p_t^*)
\end{aligned}
$$

- Based on the test of monthly S&P from Jan. 1926 to Dec. 2006 with some selected yield rate, the results contradict the above claim.
- We can see that the behaviors of dividend and share price are differently, which make the dividend yield less predictable.

Outline

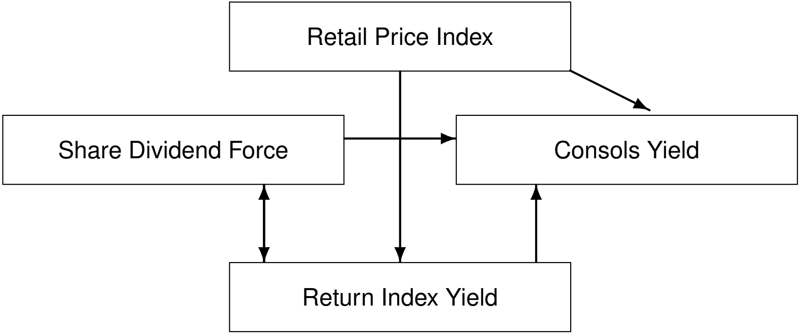
New Structure

- This new structure is constructed based on stepwise regression models on different components, their lagged values, and some rollover values.
- In the structure, we discard dividend yield and use share yield directly in the modeling.



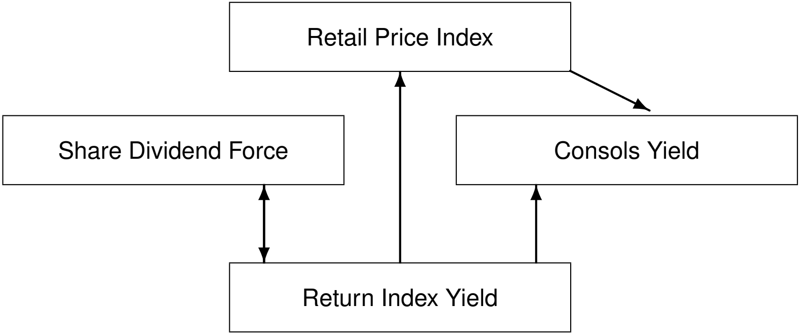
New Structure

- This structure is constructed based on the analysis of residuals of Univariate ARIMA Models.

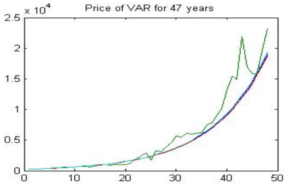


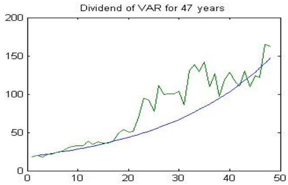
New Structure

- This structure is constructed based on the Vector Autoregressive modeling for the four economic components, with the covariance matrix for the residuals of the model of each component.



Simulation of Year TSX yield with VAR Model of New Structure





Statistics of VAR model

Statistcs	Values	Statistcs	Values
Mean	0.0898	Minimum value	-0.7535
St. dev.	0.1972	2.5 percentile	-0.2984
Skewness	-0.0252	5 percentile	-0.2351
Kurtosis	3.0202	10 percentile	-0.1622

Statistics of RSLN2 model

Statistcs	Values	Statistcs	Values
Mean	0.0906	Minimum value	-1.2167
St. dev.	0.1780	2.5 percentile	-0.2660
Skewness	0.0819	5 percentile	-0.1762
Kurtosis	6.9630	10 percentile	-0.1033

Comment: The variance of VAR model is between those of the Wilkie Model and RSLN2, while the tails of RSLN2 is thicker than that of VAR model

Outline

Conclusion

- The Wilkie Model is biased in prediction of share price, and its residuals are not white noise.

Conclusion

- The Wilkie Model is biased in prediction of share price, and its residuals are not white noise.
- In this case both VAR models and the Wilkie Model create larger variance than the univariate models. Actually in the Wilkie Model, its variance is so wide that almost everything could happen.

Conclusion

- The Wilkie Model is biased in prediction of share price, and its residuals are not white noise.
- In this case both VAR models and the Wilkie Model create larger variance than the univariate models. Actually in the Wilkie Model, its variance is so wide that almost everything could happen.
- It is better to include the series of TSX yield to predict the movement of itself. Otherwise, neither dividend yield nor dividend could predict TSX yield accurately.

Conclusion

- The Wilkie Model is biased in prediction of share price, and its residuals are not white noise.
- In this case both VAR models and the Wilkie Model create larger variance than the univariate models. Actually in the Wilkie Model, its variance is so wide that almost everything could happen.
- It is better to include the series of TSX yield to predict the movement of itself. Otherwise, neither dividend yield nor dividend could predict TSX yield accurately.
- The monthly data and the quarterly data could have different results and interpretation for modeling. In general, the continuous assumption and discrete assumption are suitable for monthly data and quarterly data respectively.

Conclusion

- The Wilkie Model is biased in prediction of share price, and its residuals are not white noise.
- In this case both VAR models and the Wilkie Model create larger variance than the univariate models. Actually in the Wilkie Model, its variance is so wide that almost everything could happen.
- It is better to include the series of TSX yield to predict the movement of itself. Otherwise, neither dividend yield nor dividend could predict TSX yield accurately.
- The monthly data and the quarterly data could have different results and interpretation for modeling. In general, the continuous assumption and discrete assumption are suitable for monthly data and quarterly data respectively.
- Even a historically stationary model of dividend yield may not be predictable for future period, because of changing parameters for ARIMA models.

Future Works

- Updating models with other methods like cointegrated VAR models, and state space method for changing model parameters.
- Specifying any measurement for the degree of stationarity of time series for different period, and testing the ergodicity of time series.
- Improving RSLN models with smoothed regimes for multiple variables, and providing better methods for testing RSLN effects
- Updating models by taking more considerations of realities as constraints of the models.

Outline

- Chan, W.S. and Wang, S. (1998). “The Wilkie Model For Retail Price Inflation Revisited”, *British Actuarial Journal*, Vol 4, III, 637–652.
- Huber, Paul (1995). “A Review of Wilkie’s Stochastic Investment Model”, Presentation to the Staple Inn Actuarial Society, March 14th.
- Hardy, M.R. (2003). *Investment Guarantees: Modelling and Risk Management for Equity-Linked Life Insurance*, Wiley (New York).
- LeRoy, Stephen F. (1996). *Handbook of Statistics 14: Statistical Methods in Finance, edited by G.S¿ Maddala and C.R. Rao*, chapter 6, Stock Price Volatility.
- Mulvey, John M. and Thorlacius, A Eric (1996). “The Towers Perrin Global Capital Market Scenario Generation System”, *CAP–Link–Proceeding AFIR, Nürnberg, German*.

Panjer, Harry H. (1998). *Financial Economics* , Oxford University Press, chapter 9, Investment Return Models.

Rambaruth, Ganeshwarsing (2003). “A Comparison of Wilkie-Type Stochastics Investment Models”, Master Thesis, The City Univeristy, Faculty of Actuarial Science and Statistics.

Sharp, Keith P. (1992). “Modelging Canadian Price and Wage Inflation”, University of Waterloo, Department of Statistics and Actuarial Scicence.

Tomson, R. J (1996). “Stochastic Investment Modeling: the Case of South Africa”, *British Actuarial Journal*, Vol 2, 765–801.

Tsay, Ruey S. (2005). *Analysis of Financial Time Series*, 2nd edition.

- Whitten, S.P. and Thomos, R.G. (1999). “A Non-linear Scochastic Model for Actuarial Use”, *British Actuarial Journal*, Vol 5, 919–953.
- Wilkie, A.D. (1986). “A Stochastic Investment Model for Actuarial Use”, *Transactions of the Faculty of Actuaries*, Vol 39, 341–403.
- Wilkie, A.D. (1995). “More on a Stoachstic Asset Model for Actuarial Use”, *British Actuarial Journal*, Vol 1, 777-945.
- Wong, Albert C.S. and Chan, Wai Sum (2005). “Mixture Gaussian Time Series Modeling of Long–Term Market Returns”, *NAAJ*, October.

Questions?

Thanks!