Fuzzy Regression and the Term Structure of Interest Rates -- A Least Squares Approach

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ABSTRACT

Recent articles by Sánchez and Gómez (2003a, 2003b, 2004) addressed the subject of fuzzy regression (FR) and the term structure of interest rates (TSIR). Their approach relied on possibilistic regression and followed the methodology of Tanaka et. al. (1982). Although possibilistic regression has been used in many applications, it has a number of limitations, not the least of which is its nebulous relation to the least-squares concept. As an alternative to possibilistic regression, this paper uses Diamond's (1988) fuzzy least square regression to investigate the TSIR.

Keywords: fuzzy linear regression, fuzzy least-squares regression, fuzzy coefficients, possibilistic regression, term structure of interest rates.

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1. INTRODUCTION

Recent articles of Sánchez and Gómez (2003a, 2003b, 2004) addressed the subject of fuzzy regression (FR) and the term structure of interest rates (TSIR). Following Tanaka et. al. (1982), their models took the general form:

$$\widetilde{Y} = \widetilde{A}_0 + \widetilde{A}_1 x_1 + \cdots + \widetilde{A}_n x_n \quad (1)$$

where $\widetilde{Y}$ is the fuzzy output, $\widetilde{A}_j$ j=1,2,..., n, is a fuzzy coefficient, and $\mathbf{x} = (x_1, \cdots, x_n)$ is an n-dimensional non-fuzzy input vector. The fuzzy components were assumed to be triangular fuzzy numbers (TFNs). Consequently, the coefficients, for example, can be characterized by a membership function (MF), $\mu_A(a)$, a representation of which is shown in Figure 1.

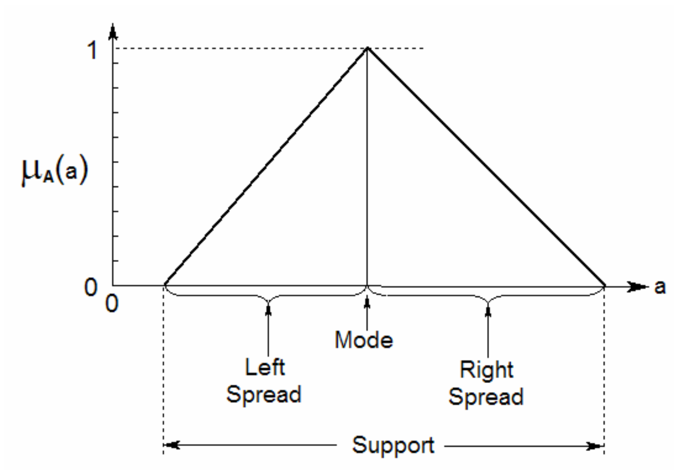


Figure 1: Fuzzy Coefficient

As indicated, the salient features of the TFN are its mode, its left and right spread, and its support. When the two spreads are equal, the TFN is known as a symmetrical TFN (STFN).

The basic idea of the Tanaka approach, often referred to as possibilistic regression, was to minimize the fuzziness of the model by minimizing the total spread of the fuzzy coefficients, subject to including all the given data. Key components of the Sánchez and Gómez methodology included constructing a discount function from a linear combination of quadratic or cubic splines, the coefficients of which were assumed to be TFNs or STFNs, and using the minimum and maximum negotiated price of fixed income assets to obtain the spreads of the dependent variable observations. Given the fuzzy discount functions, the authors provided TFN approximations[1] for the corresponding spot rates and forward rates.

[1] Since the spot rates and forward rates are nonlinear functions of the discount function, they are not TFNs even though the discount function is a TFN.

As an alternative to possibilistic regression, this paper uses Diamond's (1988) fuzzy least square regression with to investigate the TSIRs. The outline of the paper is as follows. In Section 2, we define and conceptualize the general components of fuzzy regression. The essence of the Tanaka model is explored in Section 3, including a commentary on some of its potential limitations. Section 4 discusses the fuzzy least-squares regression model as an alternative to the Tanaka model. In both the foregoing sections, the discussion is not meant to be exhaustive but, rather, is intended to point out some of the major considerations. Section 5 explores a fuzzy least square approximation of the term structure of interest rates. Section 6 compares the numerical results of Sánchez and Gómez (2004) with our findings. The paper ends with a summary of the conclusions of the study.

2. FUZZY LINEAR REGRESSION BASICS

This section provides an introduction to fuzzy linear regression. The topics addressed include the motivation for FR, the components of FR, fuzzy coefficients, the h-certain factor, and fuzzy output.

2.1 Motivation

Standard (classical) statistical linear regressions take the form

$$y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_k x_{ik} + \varepsilon_i , \qquad i = 1, 2, \cdots, m \tag{2}$$

where the dependent (response) variable, y_i, the independent (explanatory) variables, x_{ij}, and the coefficients (parameters), β_j, are crisp values, and ε_i is a crisp random error term with E(ε_i)=0, variance $\sigma^2(\varepsilon_i) = \sigma^2$, and covariance $\sigma(\varepsilon_i, \varepsilon_j) = 0 \quad \forall i, j, i \neq j$.

Although statistical regression has many applications, problems can occur in the following situations:

- Number of observations is inadequate (small data set)
- Difficulties verifying distribution assumptions
- Vagueness in the relationship between input and output variables
- Ambiguity of events or degree to which they occur
- Inaccuracy and distortion introduced by linearization

Thus, statistical regression is problematic if the data set is too small, or there is difficulty verifying that the error is normally distributed, or if there is vagueness in the relationship between the independent and dependent variables, or if there is ambiguity associated with the event or if the linearity assumption is inappropriate. These are the very situations fuzzy regression was meant to address.

2.2 The Components of Fuzzy Regression

There are two general ways (not mutually exclusive) to develop a fuzzy regression model: (1) models where the relationship of the variables is fuzzy; and (2) models where the variables themselves are fuzzy. We focus on models where the data is crisp and the relationship of the variables is fuzzy.

For any given data pair, (x_i, y_i), their role in fuzzy regression can be summarized by the fuzzy regression interval $[Y_i^L, Y_i^U]$ shown in Figure 2[2].

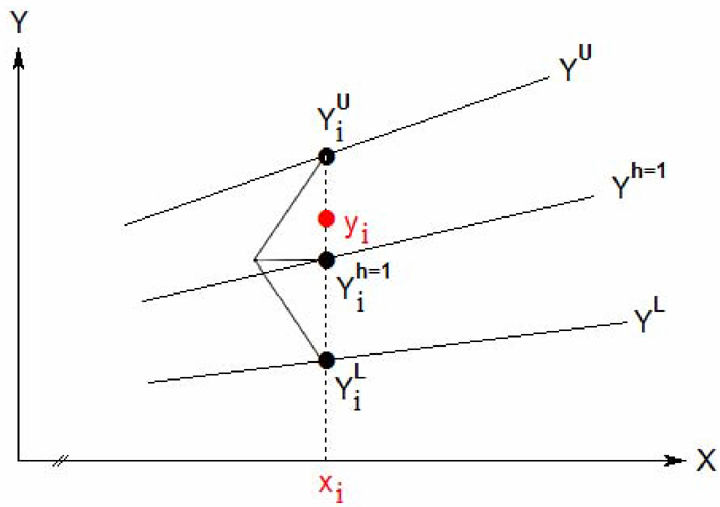


Figure 2: Fuzzy Regression Interval

$Y_i^{h=1}$ is the mode of the MF and if a STFN is assumed, $Y_i^{h=1} = \overline{Y_i} = (Y_i^U + Y_i^L)/2$. Given the parameters, $(Y_i^U, Y_i^L, Y_i^{h=1})$, which characterize the fuzzy regression model, the i-th data pair (x_i, y_i), is associated with the model parameters $(Y_i^U, Y_i^L, Y_i^{h=1})$. From a regression perspective, it is relevant to note that $Y_i^U - y_i$ and $y_i - Y_i^L$ are components of the SST, $y_i - Y_i^{h=1}$ is a component of SSE, and $Y_i^U - Y_i^{h=1}$ and $Y_i^{h=1} - Y_i^L$ are components of the SSR, as discussed by Wang and Tsaur (2000).

In possibilistic regression based on STFN, only the data points involved in determining the upper and lower bounds determine the structure of the model. The rest of the data points have no impact on the structure.

2.3 The Fuzzy Coefficients

Combining Equation (1) and Figure 1, and, for the present, restricting the discussion to

[2] Adapted from Wang and Tsaur (2000), Figure 1.

STFNs, the MF of the j-th coefficient, may be defined as:

$$\mu_{A_j}(a) = \max\left\{1 - \frac{|a - a_j|}{c_j}, 0\right\} \qquad (3)$$

where a_j is the mode and c_j is the spread, and represented as shown in Figure 3.

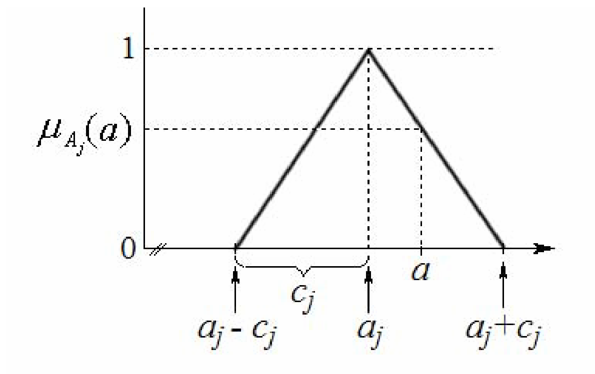


Figure 3: Symmetrical fuzzy parameters

Defining

$$\tilde{A}_j = \left\{a_j, c_j\right\}_L = \left\{\tilde{A}_j : a_j - c_j \le \tilde{A}_j \le a_j + c_j\right\}_L \qquad j = 0, 1, \cdots, n \qquad (4)$$

and restricting consideration to the case where only the coefficients are fuzzy, we can write

$$\begin{aligned} \tilde{Y}_i &= \tilde{A}_0 + \sum_{j=1}^{n} \tilde{A}_j x_{ij} \qquad (5) \\ &= (a_0, c_0)_L + \sum_{j=1}^{n} (a_j, c_j)_L x_{ij} \end{aligned}$$

This is a useful formulation because it explicitly portrays the mode and spreads of the fuzzy parameters.

2.4 Fitting the Fuzzy Regression Model

Given the foregoing, two general approaches are used to fit the fuzzy regression model:

- The possibilistic model. Minimize the fuzziness of the model by minimizing the total spreads of its fuzzy coefficients (see Figure 1), subject to including the data points of each sample within a specified feasible data interval.
- The least-squares model. Minimize the distance between the output of the model and the observed output, based on their modes and spreads.

The details of these approaches are addressed in the next two sections of this paper.

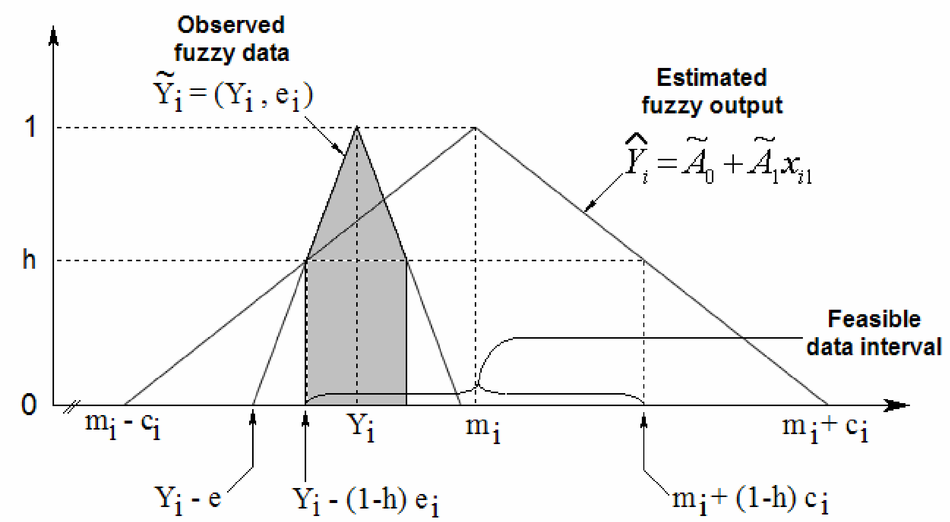
3 THE POSSIBILISTIC REGRESSION MODEL

3.1 The Model

The possibilistic regression model is optimized by minimizing the spread, subject to adequate containment of the data. The minimization of the spread takes the following form:

$$\min\left[c_0 + \sum_{j=1}^{n} c_j \,|\, x_{ij} \,|\right], \qquad c_j \geq 0 \qquad (6)$$

Putting the containment requirement together with the observed fuzzy output results in Figure 4, which shows a representation of how the estimated fuzzy output may be fitted to the observed fuzzy data.



$$\hat{Y}_i = \tilde{A}_0 + \tilde{A}_1 x_{i1}$$

$$m_i = a_0 + \sum_{j=1}^{n} a_j x_{ij} \qquad c_i = c_0 + \sum_{j=1}^{n} c_j \,|\, x_{ij} \,|$$

Figure 4: Fitting the estimated output to the observed output

For illustrative purposes, a STFN is shown, where c_i represents the spreads. Beyond that, what makes these MFs materially different from the one shown in Figure 3, is that they contain a point "h" on the y-axis, called an "h-certain factor," which, by controlling the size of the feasible data interval, extends the support of the MF[3]. In particular, as the h-factor increases, so increases the spreads, c_i.

If, as in Figure 2, the supports[4] are just sufficient to include all the data points of the sample, there would be only limited confidence in out-of-sample projection using the estimated FRM. This is resolved for FR, just as it is with statistical regression, by extending the supports.

[3] Note that the h-factor has the opposite purpose of an α-cut, in that the former is used to extend the support, while the latter is used to reduce the support.

[4] Support functions are discussed in Diamond (1988: 143) and Wünsche and Näther (2002: 47).

As indicated, the h-certain factor also can be applied to the observed output. Thus, the i-th output data might be represented by the STFN, $\widetilde{Y}_i = (y_i, e_i)$, where y_i is the mode and e_i is the spread. Here, the actual data points fall within the interval $y_i \pm (1-h)e_i$, the base of the shaded portion of the graph.

The key is that the observed fuzzy data, adjusted for the h-certain factor, is contained within the estimated fuzzy output, adjusted for the h-certain factor. Formally,

$$a_0 + \sum_{j=1}^{n} a_j x_{ij} + (1-h)\left[c_0 + \sum_{j=1}^{n} c_j | x_{ij} |\right] > y_i + (1-h)e_i \qquad (7)$$

$$a_0 + \sum_{j=1}^{n} a_j x_{ij} - (1-h)\left[c_0 + \sum_{j=1}^{n} c_j | x_{ij} |\right] < y_i - (1-h)e_i$$

$$c_j > 0, \quad i = 0,\ 1,\ \cdots,\ m, \quad j = 0,\ 1,\ \cdots,\ n$$

3.2 Criticisms of the Possibilistic Regression Model

There are a number of criticisms of the possibilistic regression model. Some of the major ones are the following:

- Tanaka et al "used linear programming techniques to develop a model superficially resembling linear regression, but it is unclear what the relation is to a least-squares concept, or that any measure of best fit by residuals is present." [Diamond (1988: 141-2)]
- The original Tanaka model was extremely sensitive to the outliers [Peters (1994)].
- There is no proper interpretation about the fuzzy regression interval [Wang and Tsaur (2000)].
- Issue of forecasting have to be addressed [Savic and Pedrycz (1991)].
- The fuzzy linear regression may tend to become multicollinear as more independent variables are collected [Kim et al (1996)].
- The solution is x_j point-of-reference dependent, in the sense that the predicted function will be very different if we first subtract the mean of the independent variables, using $(x_j - \bar{x}_i)$ instead of x_j [Hojati (2004), Bardossy (1990) and Bardossy et al (1990)].

4 A FUZZY LEAST-SQUARES REGRESSION (FLSR) MODEL

4.1 Main features of the FLSR

An obvious way to bring the FR more in line with statistical regression is to model the fuzzy regression along the same lines. In the case of a single explanatory variable, we start with the standard linear regression model: [Kao and Chyu (2003)]

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i\ , \qquad i = 1,\ 2,\ \cdots,\ m \qquad (8)$$

which, in the case most similar to the Sánchez and Gómez model, takes the form

$$\tilde{Y}_i = \tilde{A} + \tilde{B}x_i + \tilde{\varepsilon}_i \ , \qquad i = 1, 2, \cdots, m \qquad (9)$$

and requires the optimization of

$$\min_{A,B} \sum d(\widetilde{A} + \widetilde{B}x_i, \widetilde{Y}_i)^2 \qquad (10)$$

From a least squares perspective, the problem then becomes

$$\min \tilde{S} = \min \sum_{i=1}^{m} (\tilde{\varepsilon}_i)^2 = \min \sum_{i=1}^{m} (\tilde{Y}_i - \tilde{A} - \tilde{B}\, x_i)^2 \qquad (11)$$

There are a number of ways to implement FLSR, but a basic approach is FLSR using distance measures.[5] A description of this method follows.

4.2 FLSR using Distance Measures (Diamond's Approach)

Diamond (1988) was the first to implement the FLSR using distance measures and his methodology is the most commonly used. Essentially, he defined an L^2 - metric $d(.,.)^2$ between two TFNs (m_1, l_1, r_1) and (m_2, l_2, r_2) by [Diamond (1988: 143) equation (2)]

$$d\left(\langle m_1, l_1, r_1 \rangle, \langle m_2, l_2, r_2 \rangle\right)^2 = (m_1 - m_2) + \left((m_1 - l_1) - (m_2 - l_2)\right)^2 + \left((m_1 + r_1) - (m_2 + r_2)\right)^2 \qquad (12)$$

Given TFNs, it measures the distance between two fuzzy numbers based on their modes, left spread and right spread[6].

For $\tilde{Y}_i = \tilde{A} + \tilde{B}x_i + \varepsilon_i$, $i = 1, 2, \cdots, m$, the solution follows from (12), and if $\widetilde{B}$ is positive[7] (and $\widetilde{A} = (a, c_A^L, c_A^R)$, $\widetilde{B} = (b, c_B^L, c_B^R)$ and $\widetilde{Y}_i = (y_i, c_{Y_i}^L, c_{Y_i}^R)$), it takes the form:

$$d(\widetilde{A} + \widetilde{B}x_i, \widetilde{Y}_i)^2 = (a + bx_i - y_i)^2 + (a + bx_i - c_A^L - c_B^L x_i - y_i + c_{Y_i}^L)^2 + (a + bx_i + c_A^R + c_B^R x_i - y_i + c_{Y_i}^R)^2 \qquad (13)$$

A similar expression holds when $\widetilde{B}$ is negative. If the solutions exist, the parameters of $\widetilde{A}$ and $\widetilde{B}$ satisfy a system of six equations in the same number of unknowns, these equations arising from the derivatives associated with (13) being set equal to zero. Of course, this fitted

[5] An alternate basic fuzzy least-squares approach is to use compatibility measures. See Celmiņš (1987).

[6] The methods of Diamond's paper are rigorously justified by a projection-type theorem for cones on a Banach space containing the cone of triangular fuzzy numbers, where a Banach space is a normed vector space that is complete as a metric space under the metric d(x, y) = ||x-y|| induced by the norm.

[7] A triangular fuzzy number (c, a_L, a_R) is positive if $a_L \geq 0$ and negative if $a_R \leq 0$ (Shapiro, 2004:401)

model has the same general characteristics as previously shown, but now we can use the residual sum of d-squares to gauge the effectiveness of model.

The studies by Sánchez and Gómez (2003a and 2004) provide some interesting insights into the use of fuzzy regression for the analysis of the TSIRs. However, their methodology relies on possibilitic regression, which has the potential limitations mentioned in section 3.1. As an alternative, we use fuzzy least square regression (FLSR) with the distance measure defined by Diamond (1988).

5. FUZZY ESTIMATE OF THE TSIR

5.1. The problem

The input data consist of the following quantities given at a particular point in time (one session) in a fixed income market (public debt market):

- K bonds which generated a stream of cash-flows (coupon and principal)

$$\{(C_1^k, t_1^k);\ (C_2^k, t_2^k); \cdots;\ (C_{n(k)}^k, t_{n(k)}^k)\} \qquad \text{for } k = 1, \cdots, K; \qquad (14)$$

  where C_i^k is the amount of the i th cash-flow provided by the k th bond, t_i^k is its maturity (in years) and n(k) is the number of cash-flows of the k th bond.

- The minimum and maximum negotiated prices ($P_{\min}^k, P_{\max}^k$) of each bond also are given.

Assuming the bonds are non-convertible and non-callable, the price of the k th bond, P^k, is then the sum of the discounted cash flows (Sánchez and Gómez, 2003a:674)

$$P_k = \sum_{i=1}^{n(k)} C_i^k f_{t_i^k} \qquad (15)$$

where f_t is the discounted value of one dollar with a maturity of t years: i.e. $f_t = (1 + i_t)^{-t}$, and i_t is the spot rate (also called the internal rate of return (IRR)). For forecast purposes, we are interested in studying the evolution of the interest rates over time.

5.2. Motivation for fuzzy estimation of TSIR

Several studies have dealt with the modeling of interest rates. Nelson and Siegel (1987), Beekman and Shiu (1988) and later on Carriere (1999), used a four-parameter model to fit the yield curve. Local polynomial (and spline) approximation methods have also been applied (McCulloch, 1971; Vasicek and Fong, 1982; Shea, 1984). In these studies the price of the financial asset is represented by a single value. However, in practice, the price of a financial asset fluctuates within an interval, and a single-number representation can result in a lost of information. A fuzzy representation allows us to use the range of prices negotiated on the financial market during one session. Thus, this approach is more inclusive and realistic than standard econometric methods (Sánchez and Gómez, 2003b:314).

5.3. Possibilistic Estimation of the TSIR

5.3.1. Background:

Since the negotiated price P_k of the k^{th} bond oscillated between a minimum and maximum value ($P_k^{\min}$ and $P_k^{\max}$), it can be represented as a fuzzy number $\tilde{P}_k$ (Sánchez and Gómez (2003a)). In particular, TFN are used here because of their convenient properties (Dubois and Prade, 1980; Shapiro, 2004). Then,

$$\tilde{P}_k = (P_{kC}, P_{kL}, P_{kR}) \qquad k = 1, \cdots, K \tag{16}$$

where P_{kC} is the mode of $\tilde{P}_k$, and P_{kL} and P_{kR} are the left and right spreads.

McCulloch (1971) showed that the discounted function f_t in (15) can be written as a linear combination of a quadratic spline function $g_j(t)$

$$f_t = \sum_{j=0}^{m} a_j g_j(t), \tag{17}$$

where the coefficients a_j do not dependent on t. Appendix A.1 provides the explicit expression of the quadratic spline function $g_j(t)$.

A fuzzy representation of the spline decomposition (17) is as follows

$$\tilde{f}_t = \sum_{j=0}^{m} \tilde{a}_j g_j(t) \quad , \tag{18}$$

where $\tilde{f}_t = (f_{tC}, f_{tL}, f_{tR})$ and $\tilde{a}_j = (a_{jC}, a_{jL}, a_{jR})$ are fuzzy numbers with centers a_{jC} and f_{jC} and (left and right) spreads a_{jL}, a_{jR}, f_{tL}, f_{tR}.

Therefore, by combining (16) and (18), a fuzzy formulation of (15) is

$$\tilde{P}_k = (P_{kC}, P_{kL}, P_{kR}) = \sum_{i=1}^{n(k)} C_i^k \tilde{f}_{t_i^k} = \sum_{i=1}^{n(k)} C_i^k \left(\sum_{j=0}^{m} \tilde{a}_j g_j(t_i^k) \right) \tag{19}$$

With initial values $\tilde{a}_0 = (1,0,0)$, $g_0(t)=1$ and $g_j(0)=0$, $j=1,...,m$ (Sánchez and Gómez, 2004; 810), (18) and (19) become

$$\tilde{f}_t = (1,0,0) + \sum_{j=1}^{m} (a_{jC}, a_{jL}, a_{jR}) g_j(t) \quad . \tag{20}$$

$$(P_{kC}, P_{kL}, P_{kR}) = \sum_{i=1}^{n(k)} C_i^k (1,0,0) + \sum_{i=1}^{n(k)} C_i^k \sum_{j=1}^{m} (a_{jC}, a_{jL}, a_{jR}) g_j(t_i^k). \quad (21)$$

Now, denote by $\widetilde{Y}_k$ the transformation (Sánchez and Gómez, 2004)

$$\widetilde{Y}_k = (Y_{kC}, Y_{kL}, Y_{kR}) = (P_{kC}, P_{kL}, P_{kR}) - \sum_{i=1}^{n(k)} C_i^k (1,0,0) \quad (22)$$

Then the fuzzy regression model (possibilistic or least squares) reduces to solving

$$\tilde{Y}_k = \sum_{i=1}^{n(k)} C_i^k \sum_{j=1}^{m} (a_{jC}, a_{jL}, a_{jR}) g_j(t_i^k) = \sum_{j=1}^{m} (a_{jC}, a_{jL}, a_{jR}) X_j^k = \sum_{j=1}^{m} \tilde{A}_j X_j^k \quad (23)$$

where $X_j^k = \sum_{i=1}^{n(k)} C_i^k g_j(t_i^k)$, $\{k = 1, \cdots, K\}$ are known and $\widetilde{A}_j = (a_{jC}, a_{jL}, a_{jR})$ are unknown.

5.3.2. The Possibilistic Estimation of the TSIR

From (22) and (23)

$$(Y_{kC}, Y_{kL}, Y_{kR}) = (a_{1C}, a_{1L}, a_{1R}) X_1^k + \cdots + (a_{mC}, a_{mL}, a_{mR}) X_m^k, \quad (24)$$

which, in practice, is solved in two steps (Sánchez and Gómez, 2004):

- The first step consists in finding the centers a_{jC} ($j = 1, \cdots, m$) such that

  $$Y_{kC} = a_{1C} X_1^k + \cdots + a_{mC} X_m^k, \qquad \text{for } k = 1, \cdots, K. \quad (25)$$

  Appendix A.2 provides details on the technique used to solve (25). The estimated value of $(a_{1C}, \cdots, a_{mC})$ are denoted by $(\hat{a}_{1C}, \cdots, \hat{a}_{mC})$.

- The second step in solving (24) consists in computing the spreads $(a_{1L}, \cdots, a_{mL})$ and $(a_{1R}, \cdots, a_{mR})$ using the estimated centers $(\hat{a}_{1C}, \cdots, \hat{a}_{mC})$ from Step 1. Denote by $\hat{\widetilde{Y}}_k$ the estimated value of $\widetilde{Y}_k$. A fuzzy regression based on an extended version of Tanaka formula (Ishibuchi and Nii, 2001) is applied. The idea is to minimize the spread of the right hand side of (24), and simultaneously maximize the congruence of the estimate $\hat{\widetilde{Y}}_k$ with $\widetilde{Y}_k$ at the h-level. This leads to the following system to solve (Sánchez and Gómez, 2004:811)

Problem 1:

Minimize $z = \sum_{j=1}^{m} a_{jR} \sum_{k=1}^{K} \mid X_j^k \mid + \sum_{j=1}^{m} a_{jL} \sum_{k=1}^{K} \mid X_j^k \mid$ (26)

Subject to the following constraints

$$\sum_{j=1}^{m} \hat{a}_{jC} X_j^k - (1-h)\left[\sum_{j=1}^{m} a_{jL} X_j^k \right] \le Y_{kC} - Y_{kL}, \quad k = 1, \cdots, K \quad \text{(26a)}$$

$$\sum_{j=1}^{m} \hat{a}_{jC} X_j^k + (1-h)\left[\sum_{j=1}^{m} a_{jR} X_j^k \right] \ge Y_{kC} + Y_{kR}, \quad k = 1, \cdots, K \quad \text{(26b)}$$

$$\frac{\sum_{j=1}^{m} a_{jL} g_j(sP)}{1 + \sum_{j=1}^{m} \hat{a}_{jC} g_j(sP)} - \frac{\sum_{j=1}^{m} a_{jL} g_j((s+1)P)}{1 + \sum_{j=1}^{m} \hat{a}_{jC} g_j((s+1)P)} \le 0, \quad s = 1, \cdots, u-1 \quad \text{(26c)}$$

$$\frac{\sum_{j=1}^{m} a_{jR} g_j(sP)}{1 + \sum_{j=1}^{m} \hat{a}_{jC} g_j(sP)} - \frac{\sum_{j=1}^{m} a_{jR} g_j((s+1)P)}{1 + \sum_{j=1}^{m} \hat{a}_{jC} g_j((s+1)P)} \le 0, \quad s = 1, \cdots, u-1 \quad \text{(26d)}$$

$$a_{jR}, a_{jL} > 0 \quad j = 1, \cdots, m \quad \text{(26e)}$$

where P is an arbitrary periodicity (in years) and u is the future periods over which the financial rates will be determined. The conditions (26a) and (26b) ensure that each $\tilde{Y}_k$ fall within the estimated $\hat{\tilde{Y}}_k$ at level h (i.e. $\mu(\tilde{Y}_k \subseteq \hat{\tilde{Y}}_k) \ge h$). Equations (26c) and (26d) are the required conditions for the existence of the forward rates u periods ahead, and (26e) ensures that the left and right spreads are non-negative.

Once the values $\{a_{jC}, a_{jL}, a_{jR}, j = 1, \cdots, m\}$ are obtained, the discount function at time *t*, $\tilde{f}_t$, is obtained by using (20)

$$\tilde{f}_t = \sum_{j=1}^{m} \left(1 + a_{jC} g_j(t), a_{jL} g_j(t), a_{jR} g_j(t)\right).$$

The spot rate $i_t = -1 + (f_t)^{-1/t}$ is a non-linear expression of f_t. As a consequence, even though the discount function f_t is a TFN, the spot rate is not necessarily a TFN. However, a good approximation of i_t for the maturity t is given by the following fuzzy number (Dubois and Prade, 1993; Sánchez and Gómez 2004:eq. (31))

$$\tilde{i}_t = (i_{tC}, i_{tL}, i_{tR}) = \left[-1 + (f_{tC})^{-1/t}, f_{tL} / t(f_t)^{t+1/t}, f_{tR} / t(f_t)^{t+1/t} \right] \quad \text{(27).}$$

The spot rate also can be obtained using the α -cuts as described in Sánchez and Gómez (2004: 811). The forward rates for integer years, $\{{}_1\rho_t, t = 1,\cdots,u\}$, can be represented by the TFN (Sánchez and Gómez, 2004: 813)

$$ {}_1\tilde{\rho}_t = ({}_1\rho_{tC}, {}_1\rho_{tL}, {}_1\rho_{tR}) = [\ -1 + \frac{f_{t-1,C}}{f_{tC}}, \frac{f_{t-1,C}\, f_{tR} - f_{tC}\, f_{t-1,R}}{(f_{tC})^2}, \frac{f_{t-1,C}\, f_{tL} - f_{tC}\, f_{t-1,L}}{(f_{tC})^2}\ ] \quad (28) $$

Some of the potential limitations of possibilistic regression (especially its "disconnection" with the least-squares concept) can be circumvented by using fuzzy least square regression.

5.4. Estimation of the TSIR using Fuzzy Least Squares regression (FLSR)

As mentioned in §4, FLSR establishes a connection between standard least squares regression and fuzzy regression. This section shows how the Diamond (1988) version of FLSR can be used to approximate the term structure of interest rates.

In what follows, the weakest t-norm[8] T_W is used because it is shape preserving under the multiplication[9] and addition of fuzzy numbers (Mesiar, 1997; and Hong and Do, 1997). Basically, T_W replaces the t-norm $\min(a,b)$ with the t-norm

$$ T_W(x,y) = \begin{cases} \min(x,y) & if \quad \max(x,y) = 1 \\ 0 & otherwise. \end{cases} $$

For TFNs $\tilde{A} = (a, l_A, r_A)$ and $\tilde{B} = (b, l_B, r_B)$, for example, the operations reduce to (Kolesárová, 1998: Proposition 2; Hong et al. 2001:188):

- $\tilde{A} \underset{T_W}{\oplus} \tilde{B} = (a+b, \max(l_A, l_B), \max(r_A, r_B))$ (29a)
- $\tilde{A} \underset{T_W}{\otimes} \tilde{B} = (ab, \max(l_A |b|, |a| l_B), \max(r_A |b|, |a| r_B))$ (29b)

For convenience, we will use $\oplus$ and $\otimes$ to denote $\underset{T_W}{\oplus}$ and $\underset{T_W}{\otimes}$, respectively.

5.4.1. Problem formulation

A formulation of (24) with T_W − based addition gives

$$ \hat{\tilde{Y}}^k = \overset{m}{\underset{j=1}{\oplus}} \tilde{A}_j X_j^k = \tilde{A}_1 X_1^k \oplus \tilde{A}_2 X_2^k \oplus \cdots \oplus \tilde{A}_m X_m^k \quad \text{for } k = 1,\cdots,K \qquad (30) $$

[8] A triangular norm (t-norm) is a binary operation T on [0, 1], which is associative, commutative, non-decreasing and verifies T(x, 1) = x for all x ∈ [0, 1] (Zimmermann, 1996, 31).
[9] The multiplication of TFNs is an issue because it can result in a fuzzy number whose sides are drumlike.

Applying (29a-b) with $\tilde{A}_j = (a_{jC}, a_{jL}, a_{jR})$ in (30) gives

$$\hat{\tilde{Y}}^k = (a_{1C} X_1^k, a_{1L} \mid X_1^k \mid, a_{1R} \mid X_1^k \mid) \oplus \cdots \oplus (a_{mC} X_m^k, a_{mL} \mid X_m^k \mid, a_{mR} \mid X_m^k \mid), \qquad (31)$$

which, for each k=1, ..., K, reduces to

$$(Y_C^k, Y_L^k, Y_R^k) = (a_{1C} X_1^k + \cdots + a_{mC} X_m^k, \max\{a_{1L} \mid X_1^k \mid; \cdots; a_{mL} \mid X_m^k \mid\}, \max\{a_{1R} \mid X_1^k \mid; \cdots; a_{mR} \mid X_m^k \mid\})$$

Then, the fuzzy regression problem reduces to the following.

Problem 2:

$$\text{Minimize } z = \sum_{k=1}^{27} D_{LR}\left[\hat{\tilde{Y}}^k, \tilde{Y}^k\right]^2$$

$$\Leftrightarrow \text{ Minimize} \sum_{k=1}^{27} D_{LR}[\ \tilde{A}_1 X_1^k \oplus \tilde{A}_2 X_2^k \oplus \cdots \oplus \tilde{A}_m X_m^k, \tilde{Y}^k\ ]^2 \qquad (32a)$$

where the corresponding Diamond distance D_{LR} is given by (13)

$$\begin{aligned} &D_{LR}[\ \tilde{A}_1 X_1^k \oplus \tilde{A}_2 X_2^k \oplus \cdots \oplus \tilde{A}_m X_m^k, \tilde{Y}^k\ ]^2 \\ &\quad = [\ a_{1C} X_1^k + \cdots + a_{mC} X_m^k - Y_C^k\ ]^2 \\ &\quad + [\ (a_{1C} X_1^k + \cdots + a_{mC} X_m^k - \max\{a_{1L} \mid X_1^k \mid; \cdots; a_{mL} \mid X_m^k \mid\}) - (Y_C^k - Y_L^k)\ ]^2 \\ &\quad + [\ (a_{1C} X_1^k + \cdots + a_{mC} X_m^k + \max\{a_{1R} \mid X_1^k \mid; \cdots; a_{mR} \mid X_m^k \mid\}) - (Y_C^k + Y_R^k)\ ]^2 \end{aligned} \qquad (32b),$$

Equations (32a) and (32b) are solved with the same constraints [(26c)- (26d)] as used in Sánchez and Gómez (2004:811).

For j=1, ..., m, $\hat{a}_{jC}$ denotes the estimated values of a_{jC}.

Then, as shown in Appendix 4,

$$\begin{aligned} D_{LR}[\ \hat{\tilde{Y}}^k, \tilde{Y}^k\ ]^2 &= 3\sum_{j=1}^{m} \left(a_{jC} X_j^k\right)^2 + \underbrace{3\left(Y_C^k\right)^2 + (Y_L^k)^2 + (Y_R^k)^2 + 2Y_C^k(Y_R^k - Y_L^k)}_{known\ \ terms} \\ &+6\left(\sum_{j=1}^{m}\sum_{i=j+1}^{m} (a_{jC} X_j^k)(a_{iC} X_i^k)\right) - 6Y_C^k\left(\sum_{j=1}^{m} a_{jC} X_j^k\right) - 2\left(\sum_{j=1}^{m} a_{jC} X_j^k\right)(Y_R^k - Y_L^k) \\ &+\left[\max\{a_{1L} \mid X_1^k \mid, \cdots, a_{mL} \mid X_m^k \mid\}\right]^2 + \left[\max\{a_{1R} \mid X_1^k \mid, \cdots, a_{mR} \mid X_m^k \mid\}\right]^2 \\ &+2\left(\sum_{j=1}^{m} a_{jC} X_j^k\right)\left[\max\{a_{1R} \mid X_1^k \mid, \cdots, a_{mR} \mid X_m^k \mid\} - \max\{a_{1L} \mid X_1^k \mid, \cdots, a_{mL} \mid X_m^k \mid\}\right] \\ &+2(Y_C^k - Y_L^k)\max\{a_{1L} \mid X_1^k \mid, \cdots, a_{mL} \mid X_m^k \mid\} \\ &-2(Y_C^k + Y_R^k)\max\{a_{1R} \mid X_1^k \mid, \cdots, a_{mR} \mid X_m^k \mid\} \end{aligned} \qquad (33)$$

Problem 2 can be solved using nonlinear programming.

Let us denote by F_k the expression

$$F_k = D_{LR}\left[\ \hat{\tilde{Y}}^k, \tilde{Y}^k\ \right]^2 - \underbrace{3\left(Y_C^k\right)^2 - (Y_L^k)^2 - (Y_R^k)^2 - 2Y_C^k(Y_R^k - Y_L^k)}_{known\quad terms}.$$

Then, minimizing $D_{LR}\left[\ \hat{\tilde{Y}}^k, \tilde{Y}^k\ \right]^2$ reduces to minimizing F_k.

The inputs in (33) are X_i^k and $\tilde{Y}^k = (Y_C^k, Y_L^k, Y_R^k)$, for $k = \{1, \cdots, 27\}$, and the outputs are a_{jC}, a_{jL} and a_{jR} ($j = 1, \cdots, 5$).

The partial derivatives of F_k (with respect to the unknown parameters) contain the function "max", which makes it difficult to differentiate (see Appendix A.3). Therefore, nonlinear programming is used to solve Problem 2 (Nash and Sofer, 1996). A detail overview of Problem 2 solution is provided in Appendix A.3.

The steps in solving this problem can be summarized as follows, using the definitions of A_k, $p_C(k,j)$, $p_L(k,j)$, $p_R(k,j)$, $g(a_{1C}, ..., a_{mC})$, and q(x,y,k) found in Appendix 3:

Step 1: compute the known terms, for each asset $k = 1, \cdots, 27$

$$S_0^k = 3(Y_C^k)^2 + (Y_L^k)^2 + (Y_R^k)^2 + 2Y_C^k(Y_R^k - Y_L^k),\ B_k = Y_C^k - Y_L^k \text{ and } C_k = Y_C^k + Y_R^k.$$

Step 2:

Start with initial value $(a_{1C}, \cdots, a_{mC})$, $(a_{1L}, \cdots, a_{mL})$, and $(a_{1R}, \cdots, a_{mR})$

For j=1, ..., m, compute A_k, $p_C(k,j)$, $p_L(k,j)$, $p_R(k,j)$, and $g(a_{1C}, \cdots, a_{mC})$

Step 3: find the maximums;

$p_{C\max}^k = \max\{\ p_C(k,1), \cdots, p_C(k,m)\ \}$, $p_{L\max}^k = \max\{\ p_L(k,1), \cdots, p_L(k,m)\ \}$,

$p_{R\max}^k = \max\{\ p_R(k,1), \cdots, p_R(k,m)\ \}$.

Step 4: Minimize$\left[\ g(a_{1C}, \cdots, a_{mC}) + q(p_{C\max}^k, p_{L\max}^k, p_{R\max}^k)\ \right]$.

6. NUMERICAL EXAMPLE

6.1. Data and Primary Computation

In this subsection, we compare the results from the FLSR with the fuzzy regression results found by Sánchez and Gómez (2004). The data (Sánchez and Gómez 2004:813) consists of 27 bonds negotiated in the Spanish debt market on June 29, 2001 and are displayed on Table 1. The face value of each asset at maturity is 100. The cash-flows prior to maturity time are the product of the annual coupon and the face value. The matrix in Table 2 shows the streams of payments (C_i^k, t_i^k) by asset, $k = 1, \cdots, 27$. The numbers of payments corresponding to each asset are in column 2. For example, the second asset, which is a T-bill, only provides one cash-flow (face value) of 100 at expiration time 1.05. The 20th asset (bond) pays 9 cash-flows consisting of 8 payments of 4.00 prior to maturity and one payment of 100+4.00 at the time of maturity 8.59.

k	Asset	Coupon (annual) %	Number of coupons	Maturity in years	Minimum Price	Maximum Price
1	T-Bill	0.00	0	0.05	99.779	99.779
2	T-Bill	0.00	0	1.05	95.758	95.758
3	Bond	4.25	2	1.07	103.907	103.947
4	T-Bill	0.00	0	1.43	94.220	94.220
5	Bond	5.25	2	1.58	103.555	103.669
6	Strip	0.00	0	1.58	93.579	93.749
7	Bond	3.00	2	1.58	99.337	99.376
8	Strip	0.00	0	2.07	91.540	91.540
9	Bond	4.60	3	2.07	104.670	104.917
10	Bond	4.50	4	3.07	104.017	104.166
11	Bond	4.65	3	3.33	98.466	98.702
12	Bond	3.25	4	3.58	97.026	97.200
13	Bond	4.95	5	4.08	105.407	105.918
14	Bond	10.15	5	4.58	126.340	126.340
15	Bond	4.80	5	5.33	97.785	98.385
16	Bond	7.35	6	5.75	113.539	113.539
17	Bond	6.00	7	6.58	107.400	108.206
18	Strip	0.00	0	7.60	68.412	68.412
19	Bond	5.15	9	8.08	104.101	104.307
20	Bond	4.00	9	8.59	92.679	93.473
21	Bond	5.40	10	10.08	97.716	98.923
22	Bond	5.35	10	10.33	96.966	97.749
23	Bond	6.15	12	11.59	108.098	108.168
24	Strip	0.00	0	11.59	53.357	53.357
25	Bond	4.75	14	13.08	96.506	97.567
26	Bond	6.00	28	27.60	103.722	105.194
27	Bond	5.75	31	31.10	93.954	94.777

Table 1: Bonds negotiated in the Spanish debt market on June 29, 2001:
(From Sánchez and Gómez, 2004:813)

bond	#							
#	payments	---------------------- (payment, time) ----------------------						
1	1	(100,0.05)						
2	1	(100,1.05)						
3	2	(4.25,0.07)	(104.25,1.07)					
4	2	(100,1.43)						
5	2	(5.25,0.5 **8)**	(105.25,1.5 **8)**					
6	1	(100,1.58)						
7	2	(3.00,0.58)	(103,1.58)					
8	1	(100,2.07)						
9	3	(4.60,0.07)	(4.60,1.07)	(104.60,2.07)				
10	4	(4.50,0.07)	(4.50,1.07)	(4.50,2.07)	(104.50,3.07)			
11	3	(4.65,1.33)	(4.65,2.33)	(104.65,3.33)				
12	4	(3.25,0.58)	(3.25,1.58)	(3.25,2.58)	(103.25,3.58)			
13	5	(4.95,0.08)	(4.95,1.08)	**...**	(4.95,3.08)	(104.95,4.08)		
14	5	(10.15,0.58)	(10.15,1.58)	**...**	(10.15,3.58)	(110.15,4.58)		
15	5	(4.80,1.33)	(4.80,2.33)	**...**	(4.80,4.33)	(104.80,5.33)		
16	6	(7.35,0.75)	(7.35,1.75)	**...**	**...**	(7.3 5,4.75)	(107.3 5,5.75)	
17	7	(6.00,0.58)	(6.00,1.58)	**...**	**...**	(6.00,5.58)	(106.00,6.58)	
18	1	(100,7.60)						
19	9	(5.15,0.08)	(5.15,1.08)	**...**	**...**	**...**	(5.15,7.08)	(105.15,8.08)
20	9	(4.00,0.59)	(4.00,1.59)	**...**	**...**	**...**	(4.00,7.59)	(104.00,8.59)
21	10	(5.40,1.08)	(5.40,2.08)	**...**	**...**	**...**	(5.40,9.08)	(105.40,10.08)
22	10	(5.35,1.33)	(5.35,2.33)	**...**	**...**	**...**	(5.35,9.33)	(105.35,10.33)
23	12	(6.15,0.59)	(6.15,1.59)	**...**	**...**	**...**	(6.15,10.59)	(106.15,11.59)
24	1	(100,11.59)						
25	14	(4.75,0.08)	(4.75,1.08)	**...**	**...**	**...**	(4.75,12.08)	(104.75,13.08)
26	28	(6.00,0.60)	(6.00,1.60)	**...**	**...**	**...**	(6.00,26.60)	(106.00,27.60)
27	31	(5.75,1.10)	(5.75,2.10)	**...**	**...**	**...**	(5.75,30.10)	(105.75,31.10)

The first and second columns show the k^{th} bond with the number of payments.
The right columns display the pairs (payment, time of payment).

Table 2 : Matrix of cash flows.

The values $P_{kC} = (P^k_{\max} + P^k_{\min})/2$, $P_{kR} = (P^k_{\max} - P^k_{\min})/2$, $Y_{kC} = P_{kC} - \sum_{i=1}^{n(k)} C_i^k$ and $Y_{kR} = P_{kR}$ are displayed in Table 3. The number of bonds and the structure of maturities in the data lead to the choice of $m = 5$ knots in the spline approximation (17) of the discount function (Sánchez and Gómez, 2004:814). The corresponding g_j functions (17) are provided in Appendix A.1.

Then, we compute, for k=1,...,27, the following terms

$$X_j^k = C_1^k g_j(t_1^k) + C_2^k g_j(t_2^k) + \cdots + C_{n(k)}^k g_j(t_{n(k)}^k), \quad j = 1, \cdots, 5 .$$

For example, for the second asset, n(2)=1, with the cash-flow at t=1.05, which leads to the values

$$X_1^2 = 100 \times g_1(1.05) = 70.1108; \quad X_2^2 = 100 \times g_2(1.05) = 34.8892$$
$$X_3^2 = 100 \times g_3(1.05) = 0; \quad X_4^2 = 100 \times g_4(1.05) = 0$$
$$X_5^2 = 100 \times g_5(1.05) = 0.$$

Similarly, for the 20th bond, n(20)=9, and the cash-flows occur at t = 0.59, 1.59, ..., 8.59, which results in the values

$$X_1^{20} = 4.00[\ g_1(0.59) + g_1(1.59) + \cdots + g_1(7.59)\ ] + 104 \times g_1(8.59) = 106.2$$
$$X_2^{20} = 4.00[\ g_2(0.59) + g_2(1.59) + \cdots + g_2(7.59)\ ] + 104 \times g_2(8.59) = 247.343$$
$$X_3^{20} = 4.00[\ g_3(0.59) + g_3(1.59) + \cdots + g_3(7.59)\ ] + 104 \times g_3(8.59) = 431.116$$
$$X_4^{20} = 4.00[\ g_4(0.59) + g_4(1.59) + \cdots + g_4(7.59)\ ] + 104 \times g_4(8.59) = 239.6$$
$$X_5^{20} = 4.00[\ g_5(0.59) + g_5(1.59) + \cdots + g_5(7.59)\ ] + 104 \times g_5(8.59) = 0.$$

Recall that the objective is to find $\widetilde{A}_j = (a_{jC}, a_{jR})$ such that, for k=1, ..., 27,

$$(Y_{kC}, Y_{kR}) = (a_{1C}, a_{1R})X_1^k + (a_{2C}, a_{2R})X_2^k + \cdots + (a_{5C}, a_{5R})X_5^k \qquad (34)$$

where (Y_{kC}, Y_{kR}) and X_j^k are known. Table 4 displays the values of X_j^k for each asset.

k^{th} bond	P_{kC}	P_{kR}	Y_{kC}	Y_{kR}
1	99.7790	0.0000	-0.2210	0.0000
2	95.7580	0.0000	-4.2420	0.0000
3	103.9270	0.0200	-4.5730	0.0200
4	94.2200	0.0000	-5.7800	0.0000
5	103.6120	0.0570	-6.8880	0.0570
6	93.6640	0.0850	-6.3360	0.0850
7	99.3565	0.0195	-6.6435	0.0195
8	91.5400	0.0000	-8.4600	0.0000
9	104.7935	0.1235	-9.0065	0.1235
10	104.0915	0.0745	-13.9085	0.0745
11	98.5840	0.1180	-15.3660	0.1180
12	97.1130	0.0870	-15.8870	0.0870
13	105.6625	0.2555	-19.0875	0.2555
14	126.3400	0.0000	-24.4100	0.0000
15	98.0850	0.3000	-25.9150	0.3000
16	113.5390	0.0000	-30.5610	0.0000
17	107.8030	0.4030	-34.1970	0.4030
18	68.4120	0.0000	-31.5880	0.0000
19	104.2040	0.1030	-42.1460	0.1030
20	93.0760	0.3970	-42.9240	0.3970
21	98.3195	0.6035	-55.6805	0.6035
22	97.3575	0.3915	-56.1425	0.3915
23	108.1330	0.0350	-65.6670	0.0350
24	53.3570	0.0000	-46.6430	0.0000
25	97.0365	0.5305	-69.4635	0.5305
26	104.4580	0.7360	-163.5420	0.7360
27	94.3655	0.4115	-183.8845	0.4115

Table 3: Fuzzy price $\widetilde{P}_k$ and dependent variable $\widetilde{Y}_k$

Asset	X_1^k	X_2^k	X_3^k	X_4^k	X_5^k
1	4.921	0.079	0.000	0.000	0.000
2	70.111	34.889	0.000	0.000	0.000
3	74.068	37.777	0.000	0.000	0.000
4	78.288	64.712	0.000	0.000	0.000
5	85.634	83.706	0.000	0.000	0.000
6	79.000	79.000	0.000	0.000	0.000
7	82.791	81.689	0.000	0.000	0.000
8	79.000	122.664	5.336	0.000	0.000
9	86.160	129.979	5.581	0.000	0.000
10	89.560	193.860	51.796	0.000	0.000
11	89.929	203.774	71.801	0.000	0.000
12	88.242	204.299	92.500	0.000	0.000
13	94.636	217.787	146.417	0.600	0.000
14	115.881	255.246	211.770	6.000	0.000
15	97.865	228.288	263.667	23.100	0.000
16	112.237	254.203	312.709	39.200	0.000
17	110.281	252.162	365.666	80.300	0.000
18	79.000	191.500	350.973	138.500	0.000
19	111.542	258.299	433.495	193.800	0.000
20	106.199	247.343	431.116	239.600	0.000
21	121.233	282.212	484.392	418.500	3.000
22	121.159	283.730	488.252	447.300	4.500
23	135.395	312.690	534.635	608.700	17.000
24	79.000	191.500	369.000	503.900	15.600
25	127.777	298.592	523.084	754.300	41.800
26	209.897	493.695	885.055	2439.000	1101.200
27	219.398	519.534	937.863	2694.000	1609.100

Table 4: Values of X_j^k

Method1

Asset			Centers		
k	a1C	a2C	a3C	a4C	a5C
1	-0.0443	-0.0366	-0.0487	-0.0355	-0.0077
2	-0.0426	-0.0359	-0.0487	-0.0355	-0.0077
3	-0.0433	-0.0362	-0.0487	-0.0355	-0.0077
4	-0.0437	-0.0364	-0.0487	-0.0355	-0.0077
5	-0.0443	-0.0369	-0.0487	-0.0355	-0.0077
6	-0.0438	-0.0364	-0.0487	-0.0355	-0.0077
7	-0.0441	-0.0367	-0.0487	-0.0355	-0.0077
8	-0.0449	-0.0380	-0.0488	-0.0355	-0.0077
9	-0.0447	-0.0376	-0.0487	-0.0355	-0.0077
10	-0.0446	-0.0380	-0.0491	-0.0355	-0.0077
11	-0.0447	-0.0383	-0.0493	-0.0355	-0.0077
12	-0.0440	-0.0367	-0.0487	-0.0355	-0.0077
13	-0.0438	-0.0360	-0.0483	-0.0355	-0.0077
14	-0.0435	-0.0354	-0.0477	-0.0355	-0.0077
15	-0.0437	-0.0359	-0.0479	-0.0354	-0.0077
16	-0.0438	-0.0362	-0.0482	-0.0354	-0.0077
17	-0.0437	-0.0360	-0.0478	-0.0353	-0.0077
18	-0.0436	-0.0357	-0.0470	-0.0348	-0.0077
19	-0.0439	-0.0364	-0.0484	-0.0354	-0.0077
20	-0.0439	-0.0364	-0.0483	-0.0353	-0.0077
21	-0.0444	-0.0375	-0.0502	-0.0368	-0.0077
22	-0.0442	-0.0370	-0.0494	-0.0361	-0.0077
23	-0.0441	-0.0368	-0.0490	-0.0359	-0.0077
24	-0.0440	-0.0367	-0.0489	-0.0357	-0.0077
25	-0.0440	-0.0367	-0.0489	-0.0358	-0.0077
26	-0.0440	-0.0365	-0.0485	-0.0349	-0.0074
27	-0.0440	-0.0367	-0.0488	-0.0359	-0.0079
Mean	-0.0440	-0.0366	-0.0486	-0.0355	-0.0077

Method2

	-0.0438	-0.0368	-0.0486	-0.0355	-0.0078

Results by Sanchez and Gomez

	-0.0440	-0.0366	-0.0487	-0.0355	-0.0077

Table 5a: Centers values for $\widetilde{A}_C$ using Possibilistic and FLS regressions

	Tanaka					Diamond				
	$\hat{a}_{1C}$	$\hat{a}_{2C}$	$\hat{a}_{3C}$	$\hat{a}_{4C}$	$\hat{a}_{5C}$	$\hat{a}_{1C}$	$\hat{a}_{2C}$	$\hat{a}_{3C}$	$\hat{a}_{4C}$	$\hat{a}_{5C}$
$a_{jC} = \hat{a}_{jC}$	-0.0438	-0.0366	-0.0486	-0.0355	-0.0078	-0.0438	-0.0366	-0.0486	-0.0355	-0.0078
h=0.5	z=110.2									
a_{jL}	0.0009	0.0037	0.0064	0	0					
a_{jR}	0.0033	0	0.0045	0	0.0008					
a_{jL} (Diamond)						0.011	0.0047	0.0026	0.0009	0.0015
a_{jR} (Diamond)						0.0064	0.0029	0.0017	0.001	0.0016
h=0.75	z=220.5									
a_{jL}	0.0019	0.0074	0.0129	0	0					
a_{jR}	0.0065	0	0.0089	0	0.0016					

Table 5b: Centers and spreads of $\widetilde{A}_k$, Tanaka and Diamond distances.

6.2. Results and Comparison of fuzzy regressions estimates

Tables 5(a) and (b) show the estimated centers and spreads obtained from both the Tanaka (possibilistic) and Diamond (least squares) fuzzy regressions.

- For the center values a_{jC}: Both approaches produce identical estimated. Given $k_{\max} = 27$ and $m = 5$, the following values are obtained $\hat{a}_{1C} = -0.0438$, $\hat{a}_{2C} = -0.0366$, $\hat{a}_{3C} = -0.0486$, $\hat{a}_{4C} = -0.0355$ and $\hat{a}_{5C} = -0.0078$. These expected results agree with the findings in Sanchez and Gomez (2004:815).

- For the spreads: The left and right spreads (a_{jL}, a_{jR}) for the possibilistic regression are estimated for user-selected values of $h-$level (h=0.5 and h=0.75). With the least squares regression, the data determinate the values of the spreads, and there is no need for an arbitrarily chosen $h-$level.

By implementing the possibilistic regression using the Matlab software, we got the values of 110.2 (h=0.5) and 220.5 (h=0.75) for the objective function, which are close to the values of 109.62 and 219.25 obtained by Sanchez and Gomez. The values obtained for the left and right spreads agree with the results by Sanchez and Gomez.

The FLSR produces spread values that are lower than the results obtained with the possibilistic model, except for the components a_{4R} and a_{5R}.

Figure 5 displays the discount functions, the spot rates and the forward rates (for 30 years ahead) obtained from equations (20), (27), and (28).

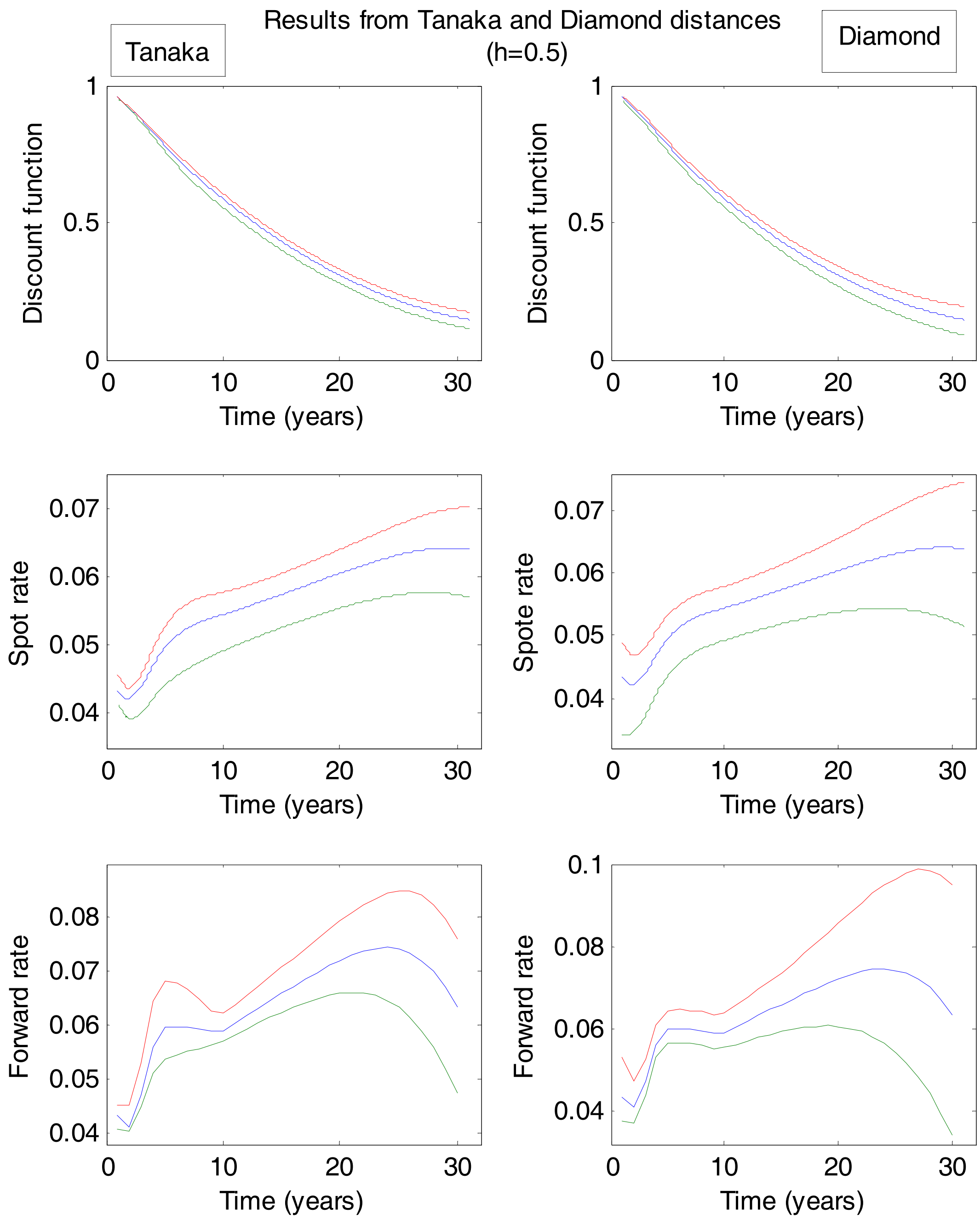


Figure 5: Discount Function, Spot and Forward rates, Tanaka and Diamond distances.

7. CONCLUSION

In this paper, we used Diamond's FLSR methodology to extend the standard econometric estimation of the TSIRs. The starting point for our analysis was the studies by Sánchez and Gómez. Those studies provide interesting insights into the use of fuzzy regression for the study of the TSIRs. However, their methodology relies on possibilitic regression, which has potential limitations, some of which can be circumvented by using FLSR techniques. While this study is still in the development stage and should be considered a work in progress, preliminary analysis suggest that both fuzzy regression models produce similar results.

APPENDICES

A.1. Spline approximation of the discount function:

The function $g_j(t)$ is based on McCulloch (1971: 29-30), as modified by Sánchez and Gómez (2004: 814). In particular, m=5, while $d_1=0$, $d_2=1.58$, $d_3=3.83$, $d_4=8.96$, and $d_5=31.1$ years, respectively. Thus,

$$g_1(t) = \begin{cases} \dfrac{-t^2}{2\times 1.58} + t & 0 \le t \le 1.58 \\ 1.58/2 & 1.58 \le t < 31.1 \end{cases}$$

$$g_j(t) = \begin{cases} 0 & 0 \le t < d_{j-1} \\ \dfrac{(t-d_{j-1})^2}{2(d_j - d_{j-1})} & d_{j-1} \le t \le d_j \\ \dfrac{-(t-d_j)^2}{2(d_{j+1}-d_j)} + (t-d_j) + \dfrac{(d_j - d_{j-1})}{2}, & d_j \le t \le d_{j+1} \\ \dfrac{(d_{j+1}-d_{j-1})}{2} & d_{j+1} \le t \le d_5 \end{cases} \qquad j = 2,3,4$$

$$g_5(t) = \begin{cases} 0 & 0 \le t \le 8.96 \\ \dfrac{(t-8.96)^2}{2(31.1-8.96)} & 8.96 \le t < 31.1 \end{cases}$$

A.2. Overview of Equation (32) solution

The equation (25) below can be solve in two ways

$$Y_{kC} = a_{1C} X_1^k + \cdots + a_{5C} X_5^k \quad \{k = 1, \cdots, 27\} .$$

Method 1: (Sánchez and Gómez)

This approach was used by Sánchez and Gómez. For each $\{k = 1, \cdots, 27\}$, ordinary least squares are used to obtain a vector $A_C^k = (a_{1C}^k, \cdots, a_{5C}^k)$. Then, the means of each vector component over k provide the vector of estimates centers $\hat{A}_C = (\hat{a}_{1C}, \cdots, \hat{a}_{mC})$.

Method 2:

Assume that the a_{jC} coefficients are the same for every k. Then, (25) has the following matrix representation,

$$\underbrace{\begin{pmatrix} Y_{1C} \\ Y_{2C} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ Y_{27,C} \end{pmatrix}}_{Y_C} = \underbrace{\begin{pmatrix} X_1^1 & X_2^1 & \cdots & \cdots & X_5^1 \\ X_1^2 & X_2^2 & \cdots & \cdots & X_5^2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ X_1^{27} & X_2^{27} & \cdots & \cdots & X_5^{27} \end{pmatrix}}_{X} \times \underbrace{\begin{pmatrix} a_{1C} \\ a_{2C} \\ a_{3C} \\ a_{4C} \\ a_{5C} \end{pmatrix}}_{A_C}$$

where Y_C and X are (27x1) and (27x5) matrices, and A_C is the (5x1) unknown vector.

A solution to this equation is obtained using MATLAB backslash (X = A\B denotes the solution to the matrix equation AX = B): $A_C = X \setminus Y_C$. As an alternative, Matlab built-in function "regress" can be used: $\text{regress}(Y_C, \text{X})$. No additional adjustment is then needed.

A.3. Overview of Problem 2 solution

The partial derivatives of F_k with respect to the unknown parameters are as follows.

$$\frac{\partial F_k}{\partial a_{jC}} = \frac{\partial}{\partial a_{jC}}\Big\{ \ 3\sum_{j=1}^{m} \left(a_{jC}X_j^k\right)^2 + 6\left(\sum_{j=1}^{m}\sum_{i=j+1}^{m} (a_{jC}X_j^k)(a_{iC}X_i^k)\right)$$

$$-6Y_C^k\left(\sum_{j=1}^{m} a_{jC}X_j^k\right) - 2\left(\sum_{j=1}^{m} a_{jC}X_j^k\right)(Y_R^k - Y_L^k) \tag{35a}$$

$$+2\left(\sum_{j=1}^{m} a_{jC}X_j^k\right)\Big[\max\{a_{1R}\mid X_1^k\mid,\cdots,a_{mR}\mid X_m^k\mid\} - \max\{a_{1L}\mid X_1^k\mid,\cdots,a_{mL}\mid X_m^k\mid\}\Big]$$

$$\frac{\partial F_k}{\partial a_{jL}} = \frac{\partial}{\partial a_{jL}}\Big\{ \ \max{}^2\{a_{1L}\mid X_1^k\mid,\cdots,a_{mL}\mid X_m^k\mid\}$$

$$-2\left(Y_L^k - Y_C^k + \sum_{j=1}^{m} a_{jC}X_j^k\right)\max\{a_{1L}\mid X_1^k\mid,\cdots,a_{mL}\mid X_m^k\mid\} \quad \Big\} \tag{35b}$$

$$\frac{\partial F_k}{\partial a_{jR}} = \frac{\partial}{\partial a_{jR}}\Big\{ \ \max{}^2\{a_{1R}\mid X_1^k\mid,\cdots,a_{mR}\mid X_m^k\mid\}$$

$$+2\left(-Y_L^k - Y_C^k + \sum_{j=1}^{m} a_{jC}X_j^k\right)\max\{a_{1R}\mid X_1^k\mid,\cdots,a_{mR}\mid X_m^k\mid\} \quad \Big\} \tag{35c}$$

These expressions contain the function "max", which makes it difficult to differentiate.

Problem 2 can be summarized as follows.

Minimize $\sum_{k=1}^{27} F_k$, where

$$F_k = 3\sum_{j=1}^{m}\left(a_{jC}X_j^k\right)^2 + 6\left(\sum_{j=1}^{m}\sum_{i=j+1}^{m}(a_{jC}X_j^k)(a_{iC}X_i^k)\right) - 6Y_C^k\left(\sum_{j=1}^{m} a_{jC}X_j^k\right) - 2Y_R^k\left(\sum_{j=1}^{m} a_{jC}X_j^k\right)$$

$$+2Y_L^k\left(\sum_{j=1}^{m} a_{jC}X_j^k\right) + \left[\max\{a_{1L}\mid X_1^k\mid,\cdots,a_{mL}\mid X_m^k\mid\}\right]^2 + \left[\max\{a_{1R}\mid X_1^k\mid,\cdots,a_{mR}\mid X_m^k\mid\}\right]^2$$

$$-2\left(\sum_{j=1}^{m} a_{jC}X_j^k\right)\left[\max\{a_{1L}\mid X_1^k\mid,\cdots,a_{mL}\mid X_m^k\mid\} - \max\{a_{1R}\mid X_1^k\mid,\cdots,a_{mR}\mid X_m^k\mid\}\right]$$

$$+2(Y_C^k - Y_L^k)\max\{a_{1L}\mid X_1^k\mid,\cdots,a_{mL}\mid X_m^k\mid\} - 2(Y_C^k + Y_R^k)\max\{a_{1R}\mid X_1^k\mid,\cdots,a_{mR}\mid X_m^k\mid\}$$

Define the functions p_C, p_L and p_R such that
$p_C(k,j) = a_{jC}\mid X_j^k\mid$, $p_L(k,j) = a_{jL}\mid X_j^k\mid$, $p_R(k,j) = a_{jR}\mid X_j^k\mid$, $j = 1,\cdots,m$.

Also define the functions g and q such that

$$g(a_{1C},\cdots,a_{mC}) = 3\sum_{j=1}^{m}\left(a_{jC}X_j^k\right)^2 + 6\left(\sum_{j=1}^{m}\sum_{i=j+1}^{m}(a_{jC}X_j^k)(a_{iC}X_i^k)\right) - \left(6Y_C^k - 2Y_L^k + 2Y_R^k\right)\left(\sum_{j=1}^{m}a_{jC}X_j^k\right)$$

$$q(x,y,k) = x^2 + y^2 - 2A_k(x-y) + 2B_k x - 2C_k y,$$

where $A_k = \sum_{j=1}^{m} a_{jC}X_j^k$, $B_k = Y_C^k - Y_L^k$ and $C_k = Y_C^k + Y_R^k$.

$x \equiv \max\{a_{1L} \mid X_1^k \mid,\cdots,a_{mL} \mid X_m^k \mid\}$ and $y \equiv \max\{a_{1R} \mid X_1^k \mid,\cdots,a_{mR} \mid X_m^k \mid\}$.

A. 4. Derivation of (33)

Each term of the right hand side of (32b) can be written as

$$\begin{aligned}&[a_{1C}X_1^k + \cdots + a_{mC}X_m^k - Y_C^k]^2 \\ &= \sum_{j=1}^{m}\left(a_{jC}X_j^k\right)^2 + \left(Y_C^k\right)^2 + 2\left(\sum_{j=1}^{m}\sum_{i=j+1}^{m}(a_{jC}X_j^k)(a_{iC}X_i^k)\right) - 2Y_C^k\left(\sum_{j=1}^{m}a_{jC}X_j^k\right)\end{aligned} \tag{36a}$$

$$\begin{aligned}&\left[\, a_{1C}X_1^k + \cdots + a_{mC}X_m^k - \max\left\{a_{1L} \mid X_1^k \mid;\cdots;a_{mL} \mid X_m^k \mid\right\} - (Y_C^k - Y_L^k)\,\right]^2 \\ &= \sum_{j=1}^{m}\left(a_{jC}X_j^k\right)^2 + \left(\max\left\{a_{1L} \mid X_1^k \mid;\cdots;a_{mL} \mid X_m^k \mid\right\}\right)^2 + \left(Y_C^k\right)^2 + \left(Y_L^k\right)^2 \\ &+ 2\left(\sum_{j=1}^{m}\sum_{i=j+1}^{m}(a_{jC}X_j^k)(a_{iC}X_i^k)\right) - 2\left(\sum_{j=1}^{m}a_{jC}X_j^k\right)\max\left\{a_{1L} \mid X_1^k \mid;\cdots;a_{mL} \mid X_m^k \mid\right\} \\ &- 2\left(\sum_{j=1}^{m}a_{jC}X_j^k\right)(Y_C^k - Y_L^k) + 2(Y_C^k - Y_L^k)\max\left\{a_{1L} \mid X_1^k \mid;\cdots;a_{mL} \mid X_m^k \mid\right\} - 2Y_C^kY_L^k\end{aligned} \tag{36b}$$

$$\begin{aligned}&\left[\, a_{1C}X_1^k + \cdots + a_{mC}X_m^k + \max\left\{a_{1R} \mid X_1^k \mid;\cdots;a_{mR} \mid X_m^k \mid\right\} - (Y_C^k + Y_R^k)\,\right]^2 \\ &= \sum_{j=1}^{m}\left(a_{jC}X_j^k\right)^2 + \left(\max\left\{a_{1R} \mid X_1^k \mid;\cdots;a_{mR} \mid X_m^k \mid\right\}\right)^2 + \left(Y_C^k\right)^2 + \left(Y_R^k\right)^2 \\ &+ 2\left(\sum_{j=1}^{m}\sum_{i=j+1}^{m}(a_{jC}X_j^k)(a_{iC}X_i^k)\right) + 2\left(\sum_{j=1}^{m}a_{jC}X_j^k\right)\max\left\{a_{1R} \mid X_1^k \mid;\cdots;a_{mR} \mid X_m^k \mid\right\} \\ &- 2\left(\sum_{j=1}^{m}a_{jC}X_j^k\right)(Y_C^k + Y_R^k) - 2(Y_C^k + Y_R^k)\max\left\{a_{1R} \mid X_1^k \mid;\cdots;a_{mR} \mid X_m^k \mid\right\} + 2Y_C^kY_R^k\end{aligned} \tag{36c}$$

Then, (33) is obtained by summing up (36a)- (36c).

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