Conceptualizing Future Lifetime as a Fuzzy Random Variable

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Abstract

A key concept in post-retirement planning is the random variable future lifetime of a life aged x, T(x). While age is an important factor in the determination of T(x), there are other relevant factors, like the state of health and the character of a life aged x. These latter factors often are encapsulated in a perceived fuzzy metric, like "less than average future lifetime." In these instances, T(x) might more appropriately be defined as a function that assigns a fuzzy subset to each possible outcome of a random event, and conceptualized as a fuzzy random variable. The purpose of this article is to presents some preliminary observations regarding this conceptualization of future lifetime as a fuzzy random variable.

Key words: future lifetime, fuzzy random variable, fuzzy metric

1. Introduction

A key concept in post-retirement planning is the random variable future lifetime of a life aged x, T(x). The analytical nature of T(x) is discussed in actuarial texts like Bowers et al (1997: 52), Dickson et al (2009: 17)[1], and Gerber (1997: 15), while its application in a post-retirement context is explored in articles such as Babbel and Merrill (2007), Brown (2004), Horneff et al (2008), Kapur and Orszag (2002), Milevsky (2004) and Young (2004).

The foregoing citations often focus on the relationship between attained age and T(x), and the implications of that relationship. However, while age is an important factor in the determination of T(x), there are other relevant factors.[2] Moreover, some of the other dominant factors, like the state of health and the character of a life aged x, are often

[1] Dickson et al (2009: 17), in contrast to other actuarial text, use T_x to denote the random variable future lifetime.

[2] Factors that would lead to a life expectancy perception with respect to an individual include such things as blood pressure, smoking status, total cholesterol/hdl, build, driving record, and family history of cancer. See Justman (2007) for a discussion of these factors.

encapsulated in a perceived fuzzy metric, like "less than average future lifetime." Thus, T(x) might more appropriately be written as $\tau = \tau(x \mid \tilde{f})$, where the tilde denotes a fuzzy parameter and $\tilde{f}$ represents a fuzzy metric other than age, in which case τ can be conceptualized as a fuzzy random variable [Shapiro (2009)]. The purpose of this article is to presents some preliminary observations regarding this conceptualization of future lifetime as a fuzzy random variable, so, for the most part, the discussion is conceptual rather than technical.

We begin with a simple statement of the problem. This is followed by a short overview of future lifetime as a random variable, where, for simplicity, we assume the Gompertz form of the force of mortality. The parameters are chosen so that the expected lifetime at age 65 is 15 years. Next, membership functions (MFs) for short, average and long future lifetime are discussed. These are followed by a discussion of future lifetime as a fuzzy random variable. The article ends with a comment on potential applications involving future lifetime as a fuzzy random variable and areas for potential refinements of the model.

2. A Statement of the Problem

To put the problem in context, consider the following situation.[3] A consultant is asked to give post-retirement financial planning advice to a new retiree. After a discussion with the retiree, he concludes that she is a standard life, and he ponders her future lifetime. To this end, he chooses from the linguistic scale $\mathbb{L}$, which is composed of the terms "short future lifetime," "medium future lifetime," and "long future lifetime." Each of these labels can be viewed as a fuzzy subset of the future lifetime scale. This information can be described by a fuzzy random variable $\tau: \Omega \to \mathbb{L}$, where each $\omega \in \Omega$ represents a new retiree, and $\tau(\omega)$ represents the label assigned to his or her future lifetime (short, medium or long).[4]

This scenario is an example of the Puri and Ralescu (1986) view of fuzzy random variables, where the fuzzy variable, a fuzzy subset of the future lifetime scale, is associated with the randomly chosen new retiree.[5]

[3] Adapted from Couso and Dubois (2009: 1072)

[4] For a discussion of the fuzziness inherent in post-retirement financial strategies, other than that related to future lifetime, see Shapiro (2011).

[5] Another scenario would be that the new retiree is subject to underlying mortality rates that are unknown. This is an example of the model of Kwakernaak (1978), who viewed a FRV as a vague perception of a crisp but unobservable RV. In this case, fuzzy sets are perceived as observation results of the underlying RV. Shapiro (2012) discusses this scenario.

3. Future Lifetime as a Random Variable

The simplest (classical) approach is to model future lifetime in terms of an n-year period certain. The period certain might be the expected lifetime, for example. The model is given a stochastic dimension by reformulating future lifetime as a random variable (RV), T(x) say, where x is the age of the individual. This can then be conceptualized as shown in Figure 1.

Figure 1: The random variable future lifetime, T(x)

Assuming the Gompertz form of the force of mortality, $\mu_x = Bc^x$, the pdf of the future lifetime distribution, for a life aged 65, takes the form shown in Figure 2.

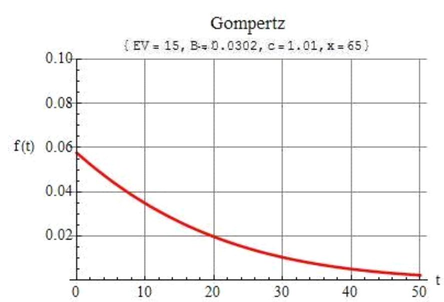

Figure 2: pdf of Future Lifetime Distribution

The cumulative distribution, assuming the same Gompertz force of mortality, is shown in Figure 3. As indicated, given our assumed parameter values, the expected lifetime at age 65, $\mathring{e}_{65}$, is 15 years.

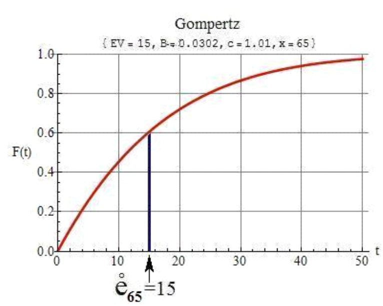

Figure 3: cdf of Future Lifetime Distribution

4. Future Lifetime as a Fuzzy Variable

Linguistic variables, which are the building blocks of fuzzy variables, may be defined (Zadeh, 1975, 1981) as variables whose values are expressed as words or sentences. Average future lifetime for a life aged x, for example, may be viewed both as a numerical value ranging over the interval [0, ω-x], where ω is the limiting age, and a linguistic variable that can take on values like short, medium, and long. Each of these linguistic values may be interpreted as a label of a fuzzy subset of the universe of discourse [0, ω-x], whose base variable is the generic numerical value future lifetime.

Assume the expected lifetime for a 65 year old is 15 years. Then, it is reasonable to perceive the average future lifetime as its expected value, and conceptualize it as a symmetrical triangular fuzzy number[6] with a left and right spread of 5 years, say, as shown in Figure 4 and written as shown in equation (1).

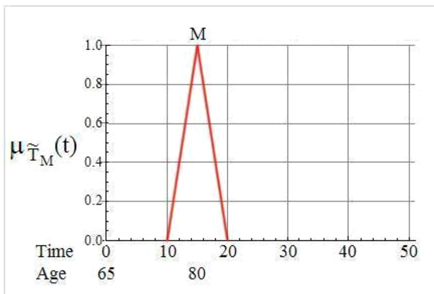

Figure 4: Membership Function for Medium (M) Future Lifetime

$$\mu_{\tilde{T}_M}(t) = \begin{cases} 0 & 0 \le t \le 10 \\ \frac{t-10}{5} & 10 < t < 15 \\ 1 & t = 15 \\ \frac{20-t}{5} & 15 < t < 20 \\ 0 & t \ge 20 \end{cases} \qquad (1)$$

As indicated, the membership function for a medium future lifetime, $\mu_{\tilde{T}_M}(t)$, assigns to each future lifetime a grade of membership (GOM) ranging between zero and one. Future lifetime, when between zero and 10, (age 65, age 75], is assigned a GOM of 0 in the set medium future lifetime, that is, defined as not at all a member of the medium future lifetime group. Future lifetime of between 10 to 15 years, (age 75, age 80), is assigned an increasing GOM, while future lifetime between 15 and 20 years, (age 80, age

[6] A symmetrical triangular fuzzy number is convenient to work with. Beyond that, however, Pedrycz (1994: 21) showed that "... under some weak assumptions, ... triangular membership functions ... comply with ... relevant optimization criteria."

85), is assigned a decreasing GOM. A future lifetime of 15 years (age 80) is assigned a GOM = 1.

Given the perceived average future lifetime, the perceived MF for short (less than average) future lifetime, $\mu_{\tilde{T}_S}(t)$, might be conceptualized as shown in Figure 5 and written as shown in equation (2).

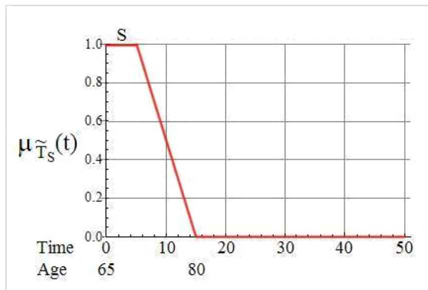

Figure 5: Membership Function for Short (S) Future Lifetime

$$\mu_{\tilde{T}_S}(t) = \begin{cases} 1 & 0 \le t < 5 \\ \frac{15-t}{10} & 5 \le t < 15 \\ 0 & t \ge 15 \end{cases} \qquad (2)$$

As indicated, $\mu_{\tilde{T}_S}(t)$ assigns to each future lifetime a grade of membership (GOM) ranging between zero and one. Future lifetime, when between zero and 5 (age 70 or less), is assigned a GOM of one in the set short future lifetime. Future lifetime of 15 years, or more, is assigned a GOM of zero, that is, defined as not at all a member of the short future lifetime group. Between those values, (5, 15), the GOM is fuzzy. If the MF has the shape depicted in Figure 5, it is characterized as reverse-S-shaped.

Finally, as indicated in Figure 6, a future lifetime of 25 years (age 90), or longer would be perceived as a long future lifetime, while a future lifetime of less than 15 years would not. In this case, the period between 15 years and 25 years is fuzzy. This MF is characterized as S-shaped.

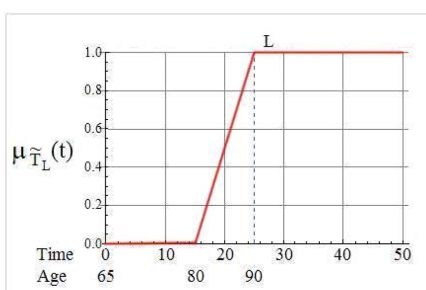

Figure 6: Membership Function for Long (L) Future Lifetime

Note that, once the basis for the linguistic average future lifetime is established, the foregoing fuzzy variables are defined (perceived) without regard for the actual distribution of T(x). In this sense, the fuzzy variable future lifetime and the random variable future lifetime are independent.

5. Future Lifetime as a Fuzzy Random Variable

Figure 7[7] shows a simple representation of future lifetime as a fuzzy random variable (FRV). Conceptually, a FRV is a RV taking fuzzy values, so we start with a probability space (Ω, $\mathcal{F}$, P) and let $\tilde{T}_1, \tilde{T}_2, ..., \tilde{T}_n$, say, be fuzzy variables.[8] Then, for each event ω_i in Ω, $\tau(\omega)$ is a FRV, where $\tau(\omega) = \tilde{T}_i$ if ω=ω_i, i=1, ..., n. In the figure, the real-valued realization is represented by $T(\omega_i)$, while the fuzzy-valued realization is represented by $\tau(\omega_i)$. Note that $T(\omega_i) \in \tau(\omega_i)$.

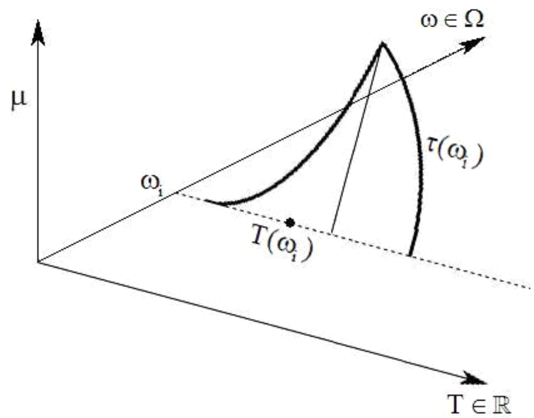

Figure 7: Representation of a FRV

As a simple example of how a FRV may be implemented, consider the fuzzy risk of a short future lifetime.[9] Following Zadeh (1968: 47), let $\tilde{P}(\tilde{T}_S)$ denote the probability of the fuzzy event short future lifetime. Then, in general,

[7] Adapted from Möller (2004: 755)

[8] Adapted from Liu (2004: 194)

[9] Adapted from Zadeh (1968) and Suresh and Mujumdar (2004).

$$\tilde{P}(\tilde{T}_S) = \int_{\mathbb{R}^n} \mu_{\tilde{T}_S}(t)\, dP$$

$$= \int_{\mathbb{R}^n} \mu_{\tilde{T}_S}(t)\; f(t)\, dt \qquad (3)$$

$$= E\{\mu_{\tilde{T}_S}\}\,,$$

where $\mathbb{R}^n$ is Euclidean n-space, $\mu_{\tilde{T}_S}(t)$ is the MF of the fuzzy event short future lifetime, t is a point in $\mathbb{R}^n$, P is a probability measure over $\mathbb{R}^n$, and f(t) is the pdf of the random variable T.[10] We see that the probability of a fuzzy event is the expectation of its MF, as noted by Zadeh (1968), and so (3) gives the expected value of the short future lifetime MF, $E\{\mu_{\tilde{T}_S}\}$.

In this instance, as we are working in the 1-D space of future lifetime defined on [0, ∞), the expected value of the short future lifetime MF is:

$$E\{\mu_{\tilde{T}_S}\} = \int_0^\infty \mu_{\tilde{T}_S}(t)\; f(t)\, dt\,. \qquad (4)$$

The results of combining the MF for short future lifetime, where $\mu_{T_S}(t)$ is given by (2), and the Gompertz pdf is conceptualized in Figure 8, where the top curve on the left is the pdf, the lower curve is the MF, and the curve on the right is the net result.

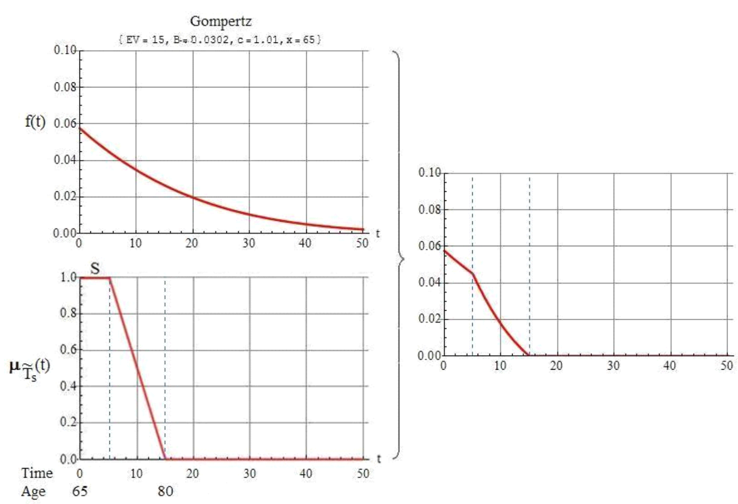

Figure 8: Combining the pdf and the MF

[10] There is not universal agreement on the appropriateness of this formulation. Singpurwalla and Booker (2004: 870-71), for example, would argue that $\tilde{P}(\tilde{T}_S)$ is not a valid probability measure. This issue is not relevant for our purposes since we focus on the expected value interpretation.

The fuzzy event long future lifetime can be modeled in a similar fashion.

5.1 Triangular-type Fuzzy Future Lifetime

The model for the fuzzy event medium future lifetime, unlike the foregoing, must accommodate triangular fuzzy numbers. To this end, consider the fuzzy triangular number $\tilde{T}_M$ in the universe $\mathbb{R}$, and its α-cut,[11] $T_{M,\alpha} = \{t \in \mathbb{R} : \mu_{\tilde{T}_M}(t) \geq \alpha\}$, as depicted in Figure 9.[12]

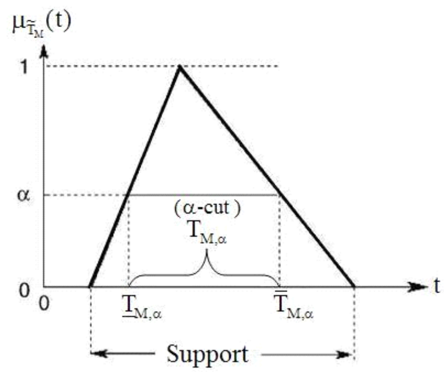

Figure 9: Alpha cut and its extreme points

As indicated in the figure, an α-cut of a fuzzy number is a closed interval and, following Kruse and Meyer (1987: 73), we denote its lower and upper extremes by $T_{M,\alpha} = [\underline{T}_{M,\alpha}, \overline{T}_{M,\alpha}]$, where

$$\underline{T}_{M,\alpha} = \inf\{t \in \mathbb{R} : \mu_{\tilde{T}_M}(t) \geq \alpha\} \qquad (5)$$
$$\overline{T}_{M,\alpha} = \sup\{t \in \mathbb{R} : \mu_{\tilde{T}_M}(t) \geq \alpha\} \qquad (6)$$

The essence of the α-cut is that it limits the domain under consideration to the set of elements with degree of membership of at least alpha, that is, the α-level set. Thus, while the support of the fuzzy set $\tilde{T}_M$ [all t such that $\mu_{\tilde{T}_M}(t) > 0$] is its entire base, its α-cut is from $\underline{T}_{M,\alpha}$ to $\overline{T}_{M,\alpha}$.

Moreover, the endpoints of the α-cuts of a triangular fuzzy random variable are real-valued random variables. [Gil et al (2006)]

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[11] The (crisp) set of elements that belong to the fuzzy set $\tilde{T}_M$ at least to the degree α.

[12] Adapted from Sinha and Gupta (2000), Figure 7.13.

Given the foregoing, the expected value of the medium future lifetime, τ_M, is a fuzzy triangular-type number where, for $\alpha \in [0,1]$,

$$E\{\tau_M\} = [E\{\mu_{\underline{T}_{M,\alpha}}\}, E\{\mu_{\overline{T}_{M,\alpha}}\}] \quad (7)$$

where

$$E\{\mu_{\underline{T}_{M,\alpha}}\} = \int_0^{\infty} \mu_{\underline{T}_{M,\alpha}}(t)\, f(t)\, dt \quad (8)$$

$$E\{\mu_{\overline{T}_{M,\alpha}}\} = \int_0^{\infty} \mu_{\overline{T}_{M,\alpha}}(t)\, f(t)\, dt \quad (9)$$

Since we are dealing with the Puri and Ralescu version of a FRV, Feng et al (2001: 487-8) counsel us that the variance should have no fuzziness. To this end, they suggest a variance computed as[13]

$$V\{\tau_M\} = \int_{\alpha=0}^{\alpha=1} \tfrac{1}{2}[V\{\mu_{\underline{T}_{M,\alpha}}\} + V\{\mu_{\overline{T}_{M,\alpha}}\}]d\alpha \quad (10)$$

where, for $\alpha \in [0,1]$, the real-valued variances have the form

$$V\{\mu_{\underline{T}_{M,\alpha}}\} = \int_0^{\infty} \left(\mu_{\underline{T}_{M,\alpha}}(t)\right)^2 f(t)\, dt - \left(E\{\mu_{\underline{T}_{M,\alpha}}\}\right)^2 \quad (11)$$

$$V\{\mu_{\overline{T}_{M,\alpha}}\} = \int_0^{\infty} \left(\mu_{\overline{T}_{M,\alpha}}(t)\right)^2 f(t)\, dt - \left(E\{\mu_{\overline{T}_{M,\alpha}}\}\right)^2 \quad (12)$$

Comment

The article presented some preliminary observations with respect to future lifetime as a fuzzy random variable. While the discussion was conceptual rather than technical, it is hoped that there was sufficient detail to give the reader a sense of the issues involved. As far as future research is concerned, extensions that can be undertaken include accommodating the Kwakernaak model, where the variance is a fuzzy variable, combining a FRV future lifetime with a FRV interest rate and evaluating various accumulation and discount models, and discrete FRV versions of the foregoing.

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[13] This is not the only possible formulation, of course. Fullér and Majlender (2002), for example, develop a general model of weighted lower and upper possibilistic mean values which they found to be consistent with Zadeh's extension principle and accepted definitions of expectation and variance in probability theory.

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