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Fuzzy Logic Modifications of the Analytic Hierarchy Process -- Some Preliminary Observations

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Abstract

The Analytic Hierarchy Process (AHP) is a theory of measurement through pair-wise comparisons that relies on judgment to derive priority scales. During its implementation, one constructs hierarchies, then makes judgments or performs measurements on pairs of elements with respect to a criterion to derive preference scales, which are then synthesized throughout the structure to select the preferred alternative.

One of the areas where the AHP finds application is in the subjective phases of risk assessment (RA). Depending on the decision-making context, however, problems can arise because decision-making often is hindered by data limitations and ambiguities, such as incomplete or unreliable data, and vague and subjective information owing to a reliance on human experts and their communication of linguistic variables. Since fuzzy logic (FL) is an effective tool in such circumstances, there has been considerable research based on adjusting the AHP for fuzziness (FAHP), and recently the focus of some of those studies has been in RA.

The literature discusses more than one FAHP model, which raises the question as to which are the prominent models and what are their characteristics. In response to this question, we examine the models underlying three of the most influential FAHP articles, based on Google Scholar citations, van Laarhoven and Pedrycz (1983), Buckley (1985) and Chang (1996). The article proceeds as follows. It begins with a brief overview of the AHP and its limitations when confronted with a fuzzy environment. This is followed with a discussion of FL modifications of the AHP. The article ends with a commentary on the situation.

Keywords: analytic hierarchy process, AHP, fuzzy logic, FAHP

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1. Introduction

The Analytic Hierarchy Process (AHP) [Saaty (1980, 1999, 2008)] is a theory of measurement through pair-wise comparisons that relies on judgment to derive priority scales. During implementation of the AHP, one constructs hierarchies, then makes judgments or performs measurements on pairs of elements with respect to a criterion to derive preference scales, which are then synthesized throughout the structure to select the preferred alternative.

One of the areas[1] where the AHP finds application is in the subjective phases of risk assessment (RA), where it is used to structure and prioritize diverse risk factors, including the judgments of experts. Depending on the decision-making context, however, problems can arise because the decision-making process often is hindered by data limitations and ambiguities, such as incomplete or unreliable data, and vague and subjective information owing to a reliance on human experts and their communication of linguistic variables. Since fuzzy logic (FL) has been shown to be an effective tool in such circumstances, there has been considerable research aimed at modeling the fuzziness in the AHP (FAHP), and recently the focus of some of that modeling has been in RA.

The examples of FAHP in RA generally relate to engineering topics. Zeng et al (2007) and Nieto-Morote and Ruz-Vila (2011), for example, presented a FAHP-based RA methodology to cope with the multitude of risks associated with complicated construction projects, where FL and the AHP were used to deal with subjective judgments and to structure the large number of risks, respectively. In a safety context, Shi et al (2012) use the FAHP to model RA associated with falling from height on construction projects, Fera and Macchiaroli (2010) used FAHP to develop a new RA model to address safety management of small and medium enterprises, and An et al (2011) used FAHP to develop a RA system for evaluating both qualitative and quantitative risk data and information associated with the safety management of railway systems. Another application area was offshore drilling, where Miri Lavasani et al (2011) used FAHP to estimate the weights required for grouping non-commensurate risk sources associated with the RA of oil and gas offshore wells, and Zhang et al (2012) use FAHP to develop a RA model of relief wells to cope with potential accidents during onshore and offshore drilling.

The literature discusses more than one FAHP model, which raises the question as to which are the prominent models and what are their characteristics. In response to this question, we examine the models underlying three of the most influential FAHP articles, based on Google Scholar citations, van Laarhoven and Pedrycz (1983), Buckley (1985) and Chang (1996). The article proceeds as follows. It begins with a brief overview of the AHP and its limitations when confronted with a fuzzy environment. This is followed with a discussion of FL modifications of the AHP. The article ends with a commentary on the situation.

[1] Surveys of other areas of AHP applications can be found in Vargas (1990), Vaidya and Kumar (2006), Subramanian and Ramanathan (2012) and Saaty and Vargas (2012).

2. Hierarchical structure and related notation

We start with a discussion of the hierarchical structure since it is key to the study of the AHP. A simple representation of a hierarchical structure is the K × n version depicted Figure 1.[2]

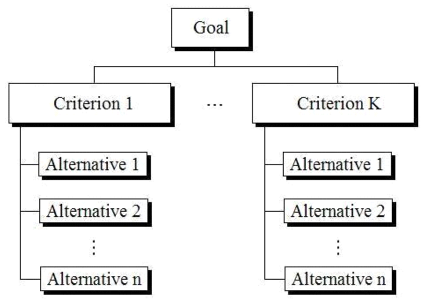

Figure 1: A simple hierarchical structure

As indicated, this hierarchy consists of three levels: [Saaty and Vargas (2012, p. 2)]

The goal of the decision at the top level,

The criteria by which the alternatives will be evaluated, in the second level, and

The alternatives, which are located in the third level.

This structure make it possible to evaluate the importance of the elements in a given level with respect to elements in a higher level.

The relationships implicit in Figure 1, and the weights associated with its components, are captured in the following notation.

A_i = the i-th alternative, i = 1, 2, ..., n
C_k = the k-th criterion, k = 1, 2, ..., K
G = the goal
$w_i^{A|G}$ = the weight associated with A_i, with respect to G
$w_{ij}^{A|G}$ = the relative preference of A_i over A_j, with respect to G

$$\equiv \frac{w_i^{A|G}}{w_j^{A|G}}, \ i, j = 1, 2, ..., n$$

$w_k^{C|G}$ = the weight associated with C_k, with respect to G
$w_{kj}^{C|G}$ = the relative preference of C_k over C_j, with respect to G

[2] Adapted from Buckley (1985b, p. 238, Fig 1).

$$\equiv \frac{w_k^{C|G}}{w_j^{C|G}}, \text{ k, j} = 1, 2, ..., K$$

$w_i^{A|C_k}$ = the weight associated with A_i, with respect to C_k

$w_{ij}^{A|C_k}$ = the relative preference of A_i over A_j, with respect to C_k

$$\equiv \frac{w_i^{A|C_k}}{w_j^{A|C_k}}, \text{ i, j} = 1, 2, ..., n; \text{ k} = 1, 2, ..., K$$

In some instances, when several decision-makers express their opinion on the relative significance of a pair of factors, there may be multiple estimates for the comparison ratios. Conversely, there may be situations where there are no estimates for certain ratios (missing data). These cases can be accommodated with an array of the form

$$\underline{w}_{ij} = (w_{ij1}, w_{ij2}, \cdots, w_{ijn_{ij}})^T, \quad (1)$$

where n_{ij}, in the last subscript, is defined as

$n_{ij} = 0$ is associated with an empty cell,
$n_{ij} = 1$ indicates a single comparison, and
$n_{ij} > 1$ indicates a cell where there are multiple comparisons.

Let "^" indicate a perceived[3] value, and following Saaty, let "a" and "c" denote the base symbols for the perceived weights associated with the alternatives and criteria, respectively. Then

$$a_{ij}^{A|G} = \hat{w}_{ij}^{A|G} \approx w_{ij}^{A|G} \quad (2)$$

$$a_{ij}^{A|C_k} = \hat{w}_{ij}^{A|C_k} \approx w_{ij}^{A|C_k} \quad (3)$$

$$c_{kj}^{C|G} = \hat{w}_{kj}^{C|G} \approx w_{kj}^{C|G} \quad (4)$$

In what follows, except for emphasis, the superscript on the c will be suppressed, since it is redundant.

3. An overview of the AHP

Given the hierarchical structure of the previous section, this section provides a brief overview of the salient features of the AHP relative to that structure.

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[3] The term "perceived" denotes a value that may be based on incomplete or unreliable data and/or vague and subjective information.

In the AHP, the decision maker carries out pairwise comparison judgments with respect to the criteria and alternatives, which are then used to develop overall priorities for ranking the alternatives. The essence of the AHP is based on the following idealized situation.[4]

As far as assumptions underlying the AHP model, it is assumed that: [Kumar and Maiti (2012, p. 9947), Adamcsek (2008, p. 7)]]

The decision-making can be modeled in a linear top-to-bottom form as a hierarchy,

The dependencies among elements can only be between the levels of the hierarchy,

The upper level in the hierarchy does not depend on the lower levels, and

The elements of a given level in a hierarchy are independent of each other.

Starting with the comparison judgments related to the criteria, the pairwise relative preference of K criteria items is modeled by a K × K preference matrix C, a representation of which is as follows:

$$C = \left[c_{kj}^{C|G} \right]_{K\times K} = \begin{bmatrix} 1 & c_{12}^{C|G} & \cdots & c_{1K}^{C|G} \\ c_{21}^{C|G} & 1 & \cdots & c_{2K}^{C|G} \\ \vdots & \vdots & \ddots & \vdots \\ c_{K1}^{C|G} & c_{K2}^{C|G} & \cdots & 1 \end{bmatrix}, \qquad (5)$$

where each cell $c_{kj}^{C|G}$ reflects how many more times, relative to the goal (G), criterion k is preferred to criterion j. To assist in this classification, Saaty (1980, Table 3-1), provided a table of relative intensities, which ranged from a minimum of 1, where the activities are "equally important", to 5, where "experience and judgement strongly favour one activity over another", to a maximum of 9, where the preference of "one activity over another is of the highest possible order of affirmation".

The construction of this matrix is a two-step process. First, the cells $c_{kj}^{C|G}$, j > k, are filled in. Then, under the assumption of a consistent preference matrix, that is, $c_{jk}^{C|G} \cdot c_{kj}^{C|G} = 1$ (reciprocal) and $c_{kj}^{C|G} = c_{ki}^{C|G} \cdot c_{ij}^{C|G}, \forall i, j, k$ (product-transitive), the cells $c_{kj}^{C|G}$, j < k, are filled in. $c_{kk}^{C|G} = 1, \forall k$.

Since C is a consistent K × K matrix, its largest eigenvalue is λ = K and there exists a corresponding eigenvector $\underline{w}^{C|G} = (w_1^{C|G}, w_2^{C|G}, \cdots, w_K^{C|G})^T$ with $c_{kj}^{C|G} \approx w_k^{C|G} / w_j^{C|G}$, $\forall k, j$, yielding the relative importance of the weights.

[4] Adapted from Adamcsek (2008) and Dubois (2011, pp. 18-19).

Given the complete preference matrix, either its principal eigenvector (discussed in §4) or one of its approximations (discussed in §5) is used as the vector of priorities.

Then, to verify that the preference matrix is sufficiently consistent, first compute λ_{max}, the principal eigenvalue, as: [Saaty and Vargas (2012, pp. 26-7)]

$$\lambda_{max} = \frac{1}{K}\sum_{k=1}^{K}\frac{(CW)_k}{w_k} = \frac{1}{K}\sum_{k=1}^{K}\frac{\sum_{j=1}^{K} c_{kj}^{C|G} w_j^{C|G}}{w_k^{C|G}}. \quad (6)$$

Given λ_{max}, the consistency index, CI, is computed as

$$CI = \frac{\lambda_{max} - K}{K-1}. \quad (7)$$

The final stage is to calculate a Consistency Ratio (CR) to measure how consistent the judgments have been relative to large samples of purely random judgments. Specifically, the consistency ratio (CR) of the preference matrix is computed as

$$CR = \frac{CI}{RI}, \quad (8)$$

where the random index, RI, is a simulated random pairwise comparisons for different size matrices, and is given in the following table.

n	1	2	3	4	5	6	7	8	9	10
RI	0.00	0.00	0.58	0.90	1.12	1.24	1.32	1.41	1.46	1.49

Source: Saaty (1980)

According to Saaty, a CR < 10% implies consistency, while if it is not less than 10% the judgments need to be revised.[5]

Similarly, the preference matrixes for the alternatives, relative to each of the criterion, $A = \left[a_{ij}^{A|C_k}\right]_{n\times n}$, k=1, 2, ..., K, are constructed and checked for consistency.

Given these two local values, the criteria preferences with respect to the goal, and the alternatives preferences with respect to the criteria, the global result, the alternatives preferences with respect to the goal, comes from their aggregation:

[5] Saaty counsel that "… improving the consistency of a judgment matrix does not necessarily improve the validity of the outcome. Validity is the goal in decision-making, not consistency, which can be successively improved by manipulating the judgments as the answer gets farther and farther from reality." [Saaty and Tran (2007)]

$$\begin{bmatrix} w_1^{A|C_1} & w_1^{A|C_2} & \cdots & w_1^{A|C_K} \\ w_2^{A|C_1} & w_1^{A|C_2} & \cdots & w_2^{A|C_K} \\ \vdots & \vdots & \ddots & \vdots \\ w_n^{A|C_1} & w_n^{A|C_2} & \cdots & w_n^{A|C_K} \end{bmatrix} \begin{bmatrix} w_1^{C|G} \\ w_2^{C|G} \\ \vdots \\ w_K^{C|G} \end{bmatrix} = \begin{bmatrix} w_1^{A|G} \\ w_2^{A|G} \\ \vdots \\ w_n^{A|G} \end{bmatrix}. \quad (9)$$

4. The Eigenvector method for determining the weights[6]

Let

$$W = [w_{ij}]_{n\times n} = \left[\frac{w_i}{w_j}\right]_{n\times n} \qquad (10)$$

be an n × n consistent pairwise comparison matrix, where $w_1, w_2, ..., w_n$ are weights and $\sum_{i=1}^{n} w_i = 1$. Then $w_{ij} = \dfrac{1}{w_{ji}}$ and $w_{ij} = \dfrac{w_i}{w_j} = \dfrac{w_i}{w_k}\dfrac{w_k}{w_j} = w_{ik}w_{kj} \;\; \forall i, j, k.$

If W is known, but $\underline{w} = (w_1, w_2, ..., w_n)^T$, the vector of weights, is not, the latter can be recovered using the eigenvalue method.

We begin by taking the matrix product of the matrix W with the vector $\underline{w}$ to obtain:

$$W\,\underline{w} = \begin{bmatrix} 1 & \frac{w_1}{w_2} & \cdots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & 1 & \cdots & \frac{w_2}{w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \cdots & 1 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} nw_1 \\ nw_2 \\ nw_3 \\ \vdots \\ nw_n \end{pmatrix} = n\,\underline{w} \qquad (11)$$

This is an eigenvalue problem of the form $W\underline{w} = \lambda\underline{w}$, where λ is an eigenvalue, which can be solved for $\underline{w}$.[7]

In this instance, since each row of W is a constant multiple of its first row, its rank is one and all its eigenvalues, save one, are equal to zero. Moreover, since the sum of the eigenvalues of a positive ($w_{ij} > 0$, ∀ i,j) matrix is equal to its trace (the sum of its diagonal elements), the non zero eigenvalue has a value of n, the order of the matrix. Since $W\underline{w} = n\underline{w}$, $\underline{w}$ is said to be the eigenvector of W corresponding to the maximum eigenvalue n. [Adamcsek (2008: 14)]

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[6] Adapted from Saaty (1980, pp. 49-51, §7-5, pp. 258-9) and Adamcsek (2008, Chapter 3).
[7] Essentially [Saaty (1980, pp. 258-9)], this reduces to the problem of finding the λ's that are the roots of |W - λ I|.

In practice, in contrast to $W=[w_{ij}]$, where $w_{ij} = w_{ik}w_{kj}$, the preference matrices, C and A, generally do not have this product-transitive characteristic, since the judgment of experts have some degree of inconsistency. In the case of the criteria, for example, w_{ij} is estimated by c_{ij}, and the eigenvalue problem for the inconsistent case is:

$$A\underline{w} = \lambda_{max}\,\underline{w}, \qquad (12)$$

where λ_{max} will be close to n (actually greater than or equal to n [Saaty (1980, p. 181)]) and the other eigenvalues will be close to zero. The estimates of the weights can be found by normalizing the eigenvector corresponding to the largest eigenvalue in the above matrix equation.

5. Alternate vectors of priorities

In the previous section, vectors of priorities were constructed from the pair-wise comparison matrix by first computing the principal eigenvector, and then normalizing it. For those instances when this approach is not feasible, Saaty (1980) suggested various approximations that can be used. [Saaty (1980: 19-21, 231-3)] Given $C=\left[c_{ij}^{C|G}\right]$, three of those methods that are used in subsequent discussions are:

(1) Normalized arithmetic mean. Sum the elements in each row and normalize by dividing each sum by the total of all the sums, so that the results add up to unity:

$$w_i = \frac{\sum_{j=1}^{K} c_{ij}}{\sum_{i=1}^{K}\sum_{j=1}^{K} c_{ij}}, \quad i = 1, 2, ..., K \qquad (13)$$

The first entry of the resulting vector is the priority of the first activity, the second of the second activity, and so on.

(2) Normalized geometric mean. Multiply the n elements in each row and take the nth root. Normalize the resulting numbers. Thus,

$$w_i = \frac{\left(\prod_{j=1}^{K} c_{ij}\right)^{1/K}}{\sum_{i=1}^{K}\left(\prod_{j=1}^{K} c_{ij}\right)^{1/K}}, \quad i = 1, 2, ..., K \qquad (14)$$

(3) Logarithmic least squares model. The vector of priority is estimated by the normalized vector that minimizes:

$$\sum_{i=1}^{K}\sum_{\substack{j=1\\j\neq i}}^{K}\left(\ln c_{ij}-\ln\left(\frac{w_i^{C|G}}{w_j^{C|G}}\right)\right)^2, \quad (15)$$

which turns out to be the same as the normalized geometric mean.

6. Perceived limitations of the AHP

Although there have been discussion regarding the broader issue of the validity of the AHP as a methodology,[8] we limit our focus here to the concerns of authors who advocate the use of fuzzy data sets as input to the AHP. The concerns include the following issues:

> It gives decision makers the opportunity to express their - essentially fuzzy - opinions in fuzzy numbers. [van Laarhoven and Pedrycz (1983)]
>
> Decision makers prefer natural language expression [Lee et al. (2013, p. 349)]
>
> It is more reliable to consider interval judgments than fixed-value judgments [Jia et al. (2013)]
>
> Crisp values are not capable of reflecting a person's vague thoughts [Kutlu and Ekmekçioğlu (2012, p. 62)]
>
> Asking for precise pairwise comparison is debatable, because these are arguably imprecisely known. [Dubois (2011, p. 19)]

7. Fuzzy AHP (FAHP) models

Given the foregoing as background, we turn now to a description of the models underlying three of the most influential FAHP articles, based on Google Scholar citations, van Laarhoven and Pedrycz (1983), Buckley (1985) and Chang (1996).[9] The articles are discussed in chronological order.

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[8] For example, Dubois (2011, p. 19) observed that, in practice, pairwise comparison data do not provide consistent matrices, and Bouyssou et al (2000, §6.3.2) expressed concern about such things as measurement issues and an absolute scale with no degrees of freedom.

[9] Although, as mentioned earlier, there have been surveys of the AHP articles, only a small portion of those studies were devoted to FAHP. While not a survey, per se, the bibliography of Dubois et al (2000) cites a number of FAHP articles.

7.1 The van Laarhoven and Pedrycz (1983) FAHP model

van Laarhoven and Pedrycz (1983) were the first to develop a FAHP. The main features of their approach were the following:

Triangular fuzzy numbers (TFNs) were used to extend the AHP to FAHP
Multiple decision-makers were accommodated
Logarithmic least squares were used to derive
 Fuzzy weights and
 Fuzzy performance scores
Approximate fuzzy multiplication was used

Following Saaty (1980, p. 231), as extended by Lootsma (1981), van Laarhoven and Pedrycz (1983) modeled their fuzzy version of the AHP using triangular membership functions and logarithmic regression.[10] The general structure of their comparison matrix for the criteria took the following form[11]:

$$\tilde{C} = \left(\underline{\tilde{c}}_{ij}\right)_{K\times K} = \begin{pmatrix} (1,1,1) & \underline{\tilde{c}}_{12} & \cdots & \underline{\tilde{c}}_{1K} \\ \underline{\tilde{c}}_{21} & (1,1,1) & \cdots & \underline{\tilde{c}}_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\tilde{c}}_{K1} & \underline{\tilde{c}}_{K2} & \cdots & (1,1,1) \end{pmatrix}, \qquad (16)$$

where $\underline{\tilde{c}}_{ij} = \left(\tilde{c}_{ij1}, \cdots, \tilde{c}_{ijn_{ij}}\right)^T$, and $\tilde{c}_{ijt} = (l_{ijt}, m_{ijt}, u_{ijt}) = \tilde{c}_{jit}^{-1} = \left(\frac{1}{u_{jit}}, \frac{1}{m_{jit}}, \frac{1}{l_{jit}}\right)$ i, j = 1, 2, ..., K, $i \neq j$, t = 0, ..., n_{ij}. $n_{ij} = 0$ is associated with an empty cell, and $n_{ij} > 1$ indicates a cell where there are multiple comparisons, which occurs when several decision-makers express their opinion on the relative significance of a pair of factors.

It follows that for the fuzzy weight vector, $\underline{w}$, the fuzzy logarithmic least squares model to be minimized is [Boender et al (1989, p. 135), Wang et al (2006, p. 3057)]

$$J = \sum_{i=1}^{K}\sum_{\substack{j=1 \\ j\neq i}}^{K}\sum_{t=1}^{n_{ij}}\left(\ln \tilde{c}_{ijt} - \ln\left(\frac{\tilde{w}_i^{C|G}}{\tilde{w}_j^{C|G}}\right)\right)^2 \qquad (17)$$

$$= \sum_{i=1}^{K}\sum_{\substack{j=1 \\ j\neq i}}^{K}\sum_{t=1}^{n_{ij}}\left((\ln c_{ijt}^L - \ln w_i^L + \ln w_j^U)^2 + (\ln c_{ijt}^M - \ln w_i^M + \ln w_j^M)^2 + (\ln c_{ijt}^U - \ln w_i^U + \ln w_j^L)^2\right)$$

where L and U denote the lower and upper extremes of a TFN and M denotes the mode.

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[10] Lootsma (1981) had shown that logarithmic regression can accommodate the case of multiple estimates for the comparison ratios and situations where there were no estimates for certain ratios (missing data). [Buckley (1985, p. 242)]

[11] Adapted from Wang et al (2006, p. 3056)

Setting $l_i = \ln w_i^L$, $m_i = \ln w_i^M, u_i = \ln w_i^U$, van Laarhoven and Pedrycz gave the normalized result as:

$$\left(\frac{\exp(l_i)}{\sum_{i=1}^{K} \exp(u_i)}, \frac{\exp(m_i)}{\sum_{i=1}^{K} \exp(m_i)}, \frac{\exp(u_i)}{\sum_{i=1}^{K} \exp(l_i)} \right), i=1, ..., K, \qquad (18)$$

which they used an estimate for w_i.

They then use their modified TFN multiplication to aggregate the local weights in order to approximate the global TFN weights for the alternatives.

7.1.1 Limitations of the Van Laarhoven and Pedrycz study

The limitations of the Van Laarhoven and Pedrycz study include:

- The formula they used for multiplication

  $$(l_1, m_1, u_1) \otimes (l_2, m_2, u_2) \sim (l_1 l_2, m_1 m_2, u_1 u_2), \quad (19)$$

  which results in a triangular fuzzy number, is only an approximation.

- A change in the priorities may cause rank reversal when replicating existing judgments on a single comparison. [Zhu (2012)]

- The methodology used to normalize the local fuzzy weights was problematic [Wang et al (2006)]

- Uncertainty of local fuzzy weights for incomplete fuzzy comparison matrices [Wang et al (2006)]

7.2 The Buckley (1985) FAHP model

The main features of Buckley (1985b) were the following:

Trapezoidal FNs were used to extend the AHP to FAHP
The geometric mean method was used to derive
  Fuzzy weights and
  Performance scores
Fuzzy multiplication and the fuzzy K-th root was used, based on
  α-cuts and interval arithmetic

Buckley's method was to substitute the fuzzy ratios $\tilde{a}_{ij}$ and $\tilde{c}_{ij}$ into the solution of the normal equations. He chose the geometric mean procedure because it resulted from the log least squares method, and he wanted a method that extends easily to fuzzy positive reciprocal matrices.

For $\tilde{C} = [\tilde{c}_{ij}^{C|G}]$, the geometric mean procedure takes the form: [Buckley (1985b, p. 237)]

$$\tilde{r}_i = (\tilde{c}_{i1} \otimes \cdots \otimes \tilde{c}_{iK})^{1/K}, \ i = 1, ..., K \qquad (20)$$

and

$$\tilde{w}_i = \tilde{r}_i \otimes (\tilde{r}_1 \oplus \cdots \oplus \tilde{r}_K)^{-1}, \ i = 1, ..., K \qquad (21)$$

where $\oplus$ and $\otimes$ represent fuzzy addition and multiplication, respectively.

Based on Buckley (1985a) and assuming the trapezoidal fuzzy number, $\tilde{c}_{ij} = (\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij})$, the increasing and decreasing portion of the MF for $\tilde{w}_i^{C|G}$ was developed as: [Buckley (1985b, p. 237)]

$$f_i(y) = \left[\prod_{j=1}^{K} ((\beta_{ij} - \alpha_{ij})y + \alpha_{ij}) \right]^{1/K}, \ i = 1, ..., K \qquad (22)$$

$$g_i(y) = \left[\prod_{j=1}^{K} ((\gamma_{ij} - \delta_{ij})y + \delta_{ij}) \right]^{1/K}, \ i = 1, ..., K \qquad (23)$$

for $0 \le y \le 1$, respectively.

Then, he defined $\alpha_i = \left[\prod_{j=1}^{K} \alpha_{ij} \right]^{1/K}$ and $\alpha = \sum_{i=1}^{K} \alpha_i$.
Similarly, he defined β_i and β, γ_i and γ, δ_i and δ.

Finally, let

$$f(y) = \sum_{i=1}^{K} f_i(y), \qquad g(y) = \sum_{i=1}^{K} g_i(y). \qquad (24)$$

Then, the fuzzy weights $\tilde{w}_i^{C|G}$ are determined by

$$\left(\frac{\alpha_i}{\delta}, \frac{\beta_i}{\gamma}, \frac{\gamma_i}{\beta}, \frac{\delta_i}{\alpha}\right),$$

where the graph of the MF for $\tilde{w}_i^{C|G}$ is

zero to the left of $\alpha_i\,\delta^{-1}$,

$x = f_i(y)/g(y)$ on the interval $[\alpha_i\,\delta^{-1}, \beta_i\,\gamma^{-1}]$,

a horizontal line from $(\beta_i\,\gamma^{-1}, 1)$ to $(\gamma_i\,\beta^{-1}, 1)$,

$x = g_i(y)/f(y)$ on the interval $[\gamma_i\,\beta^{-1}, \delta_i\,\alpha^{-1}]$, and

zero to the right of $\delta_i\,\alpha^{-1}$.

If necessary, $\tilde{w}_i^{C|G}$ can then be multiplied by a normalizing constant so that its support lies in the interval [0, 1].

Similarly, the weights, $\tilde{w}_{ij}^{A|C_k}$ and $\tilde{w}_i^{A|C_k}$, k = 1, ..., K, can be developed for the alternatives.

Then, the final fuzzy weights for the alternatives, relative to the goal, is:

$$\tilde{w}_i^{A|G} = (\tilde{w}_i^{A|C_1} \otimes \tilde{w}_1^{C|G}) \oplus \cdots \oplus (\tilde{w}_i^{A|C_K} \otimes \tilde{w}_K^{C|G}). \qquad (25)$$

These values can now be normalized.

Buckley (1985b, p. 240-1) extends this analysis to the case involving multiple experts.

7.2.1 Limitations of the Buckley study

If the positive, reciprocal matrix is perfectly consistent, then the geometric row mean procedure gives the same weights as the eigenvector method, which was Saaty's original method. However, if there is not perfect consistency, the geometric row procedure can give different weights compared to the eigenvector method. [Csutora and Buckley (2001)]

7.3 The Chang (1996) FAHP model

The main features of Chang (1996) approach were the following:

TFNs were used to extend the AHP to FAHP
Arithmetic means were used to determine the priority vector
The final ranking was done using crisp numbers

Chang (1996) used the arithmetic mean algorithm to find fuzzy priorities for the comparison matrices, whose elements were represented by triangular fuzzy numbers.

Given the criteria comparison matrix, whose elements were TFNs, $\tilde{c}_{ij}$, he applied the fuzzy counterpart of the arithmetic means, which he interpreted to be:

$$\tilde{S}_i = \sum_{j=1}^{K} \tilde{c}_{ij} \otimes \left[\sum_{k=1}^{K} \sum_{j=1}^{K} \tilde{c}_{kj} \right]^{-1} \quad (26)$$

and which he called the fuzzy synthetic extent with respect to the i-th object.[12]

He went on to interpret this value as

$$\begin{aligned} \tilde{S}_i &= \left(\frac{\sum_{j=1}^{K} l_{ij}}{\sum_{k=1}^{K} \sum_{j=1}^{K} u_{kj}}, \frac{\sum_{j=1}^{K} m_{ij}}{\sum_{k=1}^{K} \sum_{j=1}^{K} m_{kj}}, \frac{\sum_{j=1}^{K} u_{ij}}{\sum_{k=1}^{K} \sum_{j=1}^{K} l_{kj}} \right) \qquad (27) \\ &= (l_i, m_i, u_i) \;\; i = 1, \cdots, K. \end{aligned}$$

The normalized row sums $\tilde{S}_i$ (i = 1, ..., K) are then compared using the degree of possibility values: [Calabrese et al (2013, p. 3749)]

$$V(\tilde{S}_i \geq \tilde{S}_j) = \begin{cases} 1 & \text{if } m_i \geq m_j \\ \frac{u_i - l_j}{(u_i - m_i) - (m_j - l_j)} & \text{if } l_j \leq u_i \;\; i, j = 1, \cdots, K, j \neq i \\ 0 & \text{otherwise} \end{cases} \quad (28)$$

and the relative crisp weight of each item i is calculated by normalizing the degree of possibility values:

$$w_i = \frac{V(\tilde{S}_i \geq \tilde{S}_j \mid j = 1, 2, ..., K, j \neq i)}{\sum_{k=1}^{K} V(\tilde{S}_k \geq \tilde{S}_j \mid j = 1, 2, ..., K, j \neq k)}, i = 1, 2, ..., K \quad (29)$$

where

$$V(\tilde{S}_i \geq \tilde{S}_j \mid j = 1, 2, ..., K, j \neq i) = \min_{j \in (1, \cdots, K) \, j \neq i} V(\tilde{S}_i \geq \tilde{S}_j) \;\; i = 1, 2, ..., K \quad (30)$$

The foregoing formula is used to compute the local crisp weights for the criteria and alternatives, and then the standard aggregation formula for the classical AHP is used to compute the global weights for the alternatives.

[12] The term "extent analysis" refers to an analysis of the extent to which an object satisfies a goal.

7.3.1 Limitations of the Chang study

The limitations of the Chang (1996) study include:

- The normalization formula does not take into account constraints derived from the AHP method [Enea and Piazza (2004)]

- The method could lead to a wrong decision, because it may assign zero weights to some items (criteria, sub-criteria or alternatives), excluding them from the decision analysis. [Wang et al. (2008)]

8. Comment

The purpose of this article has been to present some preliminary observations regarding FL modifications of the AHP. To this end, we presented an overview of a hierarchical structure, the salient features of the AHP, the eigenvector method for determining the weights, alternate vectors of priorities, the perceived limitations of the AHP, and the dominant features of the three most commonly use FAHP models. In so far as the latter, a number of limitations of these models were mentioned; these will be addressed in a subsequent article. In addition, since this study is a part of a larger study that deals with FL applications in RA, examples of FAHP applications in RA were mentioned. Of course, the inquiry was far from exhaustive. Nonetheless, to the extent this article provides some background and an impetus for further study, it will have served its purpose.

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