QFI Quantitative Finance Exam

Exam QFIQF

Date: Thursday, October 26, 2023

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has 10 questions numbered 1 through 10 with a total of 70 points.

   The points for each question are indicated at the beginning of the question.

2. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions provided in this document.

3. Each question part or subpart should be answered either in the Excel document or the paper provided as directed. Graders will only look at the work as indicated.

4. In the Excel document, answers should be entered in the box marked ANSWER. The box will expand as lines of text are added. There is no need to use special characters or subscripts (though they may be used). For example, β_1 can be typed as beta_1 (and ^ used to indicate a superscript).

5. Prior to uploading your Excel files, each file should be saved and renamed with your five-digit candidate number in the filename.

6. The Excel file that contain your answers must be uploaded before the five-minute upload period expires.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, hand in all your written-answer sheets to the Prometric center staff. Be sure to hand in all your answer sheets because they cannot be accepted later.

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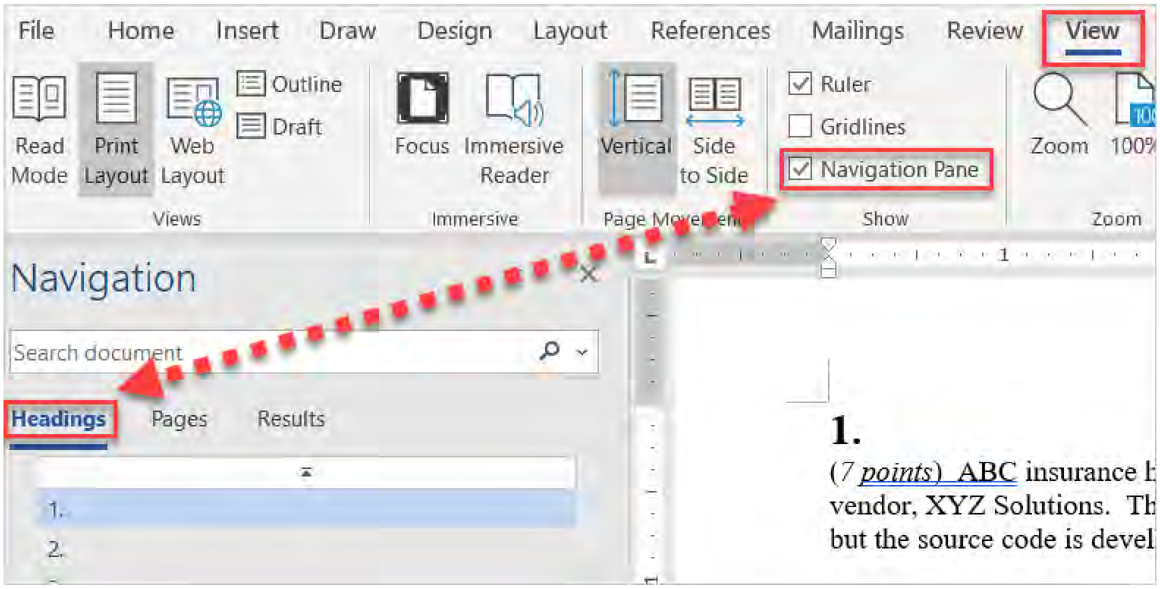
Navigation Instructions

Open the Navigation Pane to jump to questions.

Press Ctrl+F, or click View > Navigation Pane:



The responses for all parts of this question are required on the paper provided to you.

1.

(*7 points*) Your company is exploring adding barrier options to its portfolio.
Let S be the price of the underlying non-dividend paying stock, which follows a geometric Brownian motion process under the risk-neutral measure.

$$\frac{dS}{S} = rdt + \sigma dW$$

where r and $\sigma > 0$ are constant, and W is a standard Wiener process.

In addition, let $C(S, K, t)$ be the Black-Scholes price of a vanilla call at time t on S, with strike price K, expiring at time T.

Given constant α and B, define

$$V(S,t) = S^{\alpha} C\left(\frac{B^2}{S}, K, t\right).$$

(a) (*1 point*) Derive $\frac{\partial V}{\partial t}$ and $\frac{\partial V}{\partial S}$ in terms of the partial derivatives of the vanilla call.

(b) (*1 point*) Show that $\frac{\partial^2 V}{\partial S^2} = \alpha(\alpha - 1)S^{\alpha-2}C - B^2(2\alpha - 2)S^{\alpha-3}\frac{\partial C}{\partial S} + B^4 S^{\alpha-4}\frac{\partial^2 C}{\partial S^2}$

(c) (*2.5 points*) Determine the value of α such that $V(S,t)$ satisfies the Black-Scholes PDE.

You are given that the price of a down-and-in call, with barrier $B < K$, is given by $\left(\frac{S}{B}\right)^{\alpha} C\left(\frac{B^2}{S}, K, t\right)$ where α is the value found in part (c).

(d) (*2.5 points*)

(i) Describe the payoff for both a down-and-in call and a down-and-out call, each with no rebate.

(ii) Derive the formula for a down-and-out call option with respect to a vanilla call and down-and-in call with the same parameters.

(iii) Explain why the response in part (ii) implies that the down-and-out call option price also satisfies the Black-Scholes PDE.

The responses for all parts of this question are required on the paper provided to you.

2.

(*7 points*) Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space and let $W(t)$ be a standard Brownian motion with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

(a) (*2 points*) Evaluate the following expressions for $0 < s < t < u$:

(i) $E^{\mathbb{Q}}(W(s)\, W(t) W(u))$

(ii) $E^{\mathbb{Q}}(W(t) W(u) \mid \mathcal{F}_s)$

Consider the following processes:

- $V(t)$, another $\mathbb{Q}$-Brownian motion that is independent of $W(t)$ and
- $X(t)$, defined as $(V(t))^2 W(t) - \int_0^t W(s) ds$.

(b) (*1.5 points*) Determine whether $X(t)$ is a martingale under $\mathbb{Q}$ using Ito's lemma.

(c) (*3.5 points*) Determine whether $X(t)$ is a martingale under $\mathbb{Q}$ using the definition of a martingale.

The responses for all parts of this question are required on the paper provided to you.

3.

(*6 points*) Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let S_0 be the spot price of a commodity ABC. Assume that for time $t > 0$, ABC's price S_t is modeled using the following Ornstein-Uhlenbeck process:

$$dS_t = \theta(\mu - S_t)\,dt + \sigma\,dW_t,$$

where θ, μ, σ are positive constants and W_t is a $\mathbb{P}$-standard Wiener process with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

(a) (*1 point*) Show that the solution to the stochastic differential equation above is:

$$S_t = S_0 e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right) + \sigma \int_0^t e^{-\theta(t-u)} dW_u.$$

(b) (*2 points*) Derive

(i) a lower bound for $E^{\mathbb{P}}[S_t^2]$ using Jensen's inequality.

(ii) the exact value of $E^{\mathbb{P}}[S_t^2]$ using Ito isometry.

Assume that the continuously compounded risk-free rate is a constant r.

Your colleague claims that the discounted price $S_t e^{-rt}$ can be turned into a martingale with respect to a suitably defined risk-neutral measure $\mathbb{Q}$.

(c) (*1.5 points*) Determine the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$ that validates your colleague's claim.

Now consider a digital option on ABC that expires at time $T = 1$ and has payoff $\mathbb{I}_{S_1 > K}$.

(d) (*1.5 points*) Compute the fair value of this option using risk-neutral valuation.

The responses for all parts of this question are required on the paper provided to you.

4.

(*7 points*) You are given the following Hull-White model, which is calibrated to the current market bond prices, to value at time 0, 1 year and 2 months European call option on a zero-coupon bond that will mature in 2.5 years.

$$dr_t = (\theta_t - \gamma^* r_t)dt + \sigma dX_t,$$

where γ^* and σ are positive constants, and X_t is a standard Brownian motion.

The price of a zero-coupon bond with \$1 principal at time t with maturity date T is given by $Z(r,t;T) = e^{A(t;T)-B(t;T)r}$.

Let $f(t,T) = -\frac{\partial lnZ(r,t;T)}{\partial T}$ be the continuously compounded instantaneous forward rate.

You are given the following:

- $f(0,t) = c + mt$ for $t \geq 0$, where c and m are positive constants.
- $B(t;T) = \frac{1-e^{-\gamma^*(T-t)}}{\gamma^*}$ where $t \leq T$.

(a) (*0.5 points*) Demonstrate that $\theta_t = m + \gamma^* c + \gamma^* mt + \frac{\sigma^2}{2\gamma^*}(1 - \mathrm{e}^{-2\gamma^* t})$

(b) (*2.5 points*) Show that:

(i) $A(0;T) = \frac{c}{\gamma^*}(1 - \mathrm{e}^{-\gamma^* T}) - cT - \frac{mT^2}{2}$

(ii) $Z(r_0, 0; T) = e^{\frac{-mT^2}{2} - r_o T}$

In addition, you are given the following:

- $\theta_{\left(\frac{1}{2\gamma^*}\right)} - \theta_0 = 0.084$
- $m = 0.0862$
- $\sigma = 0.2$
- r_o = Initial short-term interest rate = 4%

(c) (*1 point*) Calculate γ^*.

4. Continued

(d) (*3 points*) Calculate the price at t=0 of the1 year and 2 months European call with strike price of $90, written on zero-coupon bond with face value $100 and maturity of 2.5 years.

The responses for all parts of this question are required on the paper provided to you.

5.

(*5 points*) You are simulating real-world interest rate paths on every trading day from the following Vasicek model with parameters:

$$dr_t = \gamma(\bar{r} - r_t)dt + \sigma dX_t$$

where $\bar{r} = 5\%, \gamma = 0.3, \sigma = 0.06$, and X_t is a standard Brownian motion.

Assume that there are 252 trading days in a year.

You just simulated r_t as 1% for trading day t.

(a) (*1 point*) Calculate the probability of simulating a negative interest rate for the next trading day.

You generated a random number -1.96 from the standard normal distribution.

(b) (*2 points*) Calculate the simulated rate for the next trading day using

(i) the Euler-Maruyama discretization method.

(ii) the transition density method.

(c) (*2 points*) Compare and contrast the Euler-Maruyama discretization method and the transition density method for simulating interest rate paths in general and in this particular case for Vasicek model.

6.

(*9 points*) Under the Vasicek interest rate model, the fundamental pricing equation is as follows:

$$\frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial r}\gamma(\bar{r} - r) + \frac{1}{2}\frac{\partial^2 Z}{\partial r^2}\sigma^2 = rZ$$

where γ, $\bar{r}$, σ are constants, r is the short rate, and Z is an interest rate security.
Let Z_t represent the price at time *t* of a zero-coupon bond maturing at time T. You are given that Z_t has the form:

$$Z_t = e^{A(t,T) - B(t,T) r_t}$$

where $A(t,T)$ and $B(t,T)$ satisfy the following ordinary differential equations (ODEs).

$$\frac{dA(t,T)}{dt} = B(t,T)\gamma\bar{r} - \frac{1}{2}B(t,T)^2\sigma^2$$
$$\frac{dB(t,T)}{dt} = B(t,T)\gamma - 1$$

(a) (*3 points*) Derive the expression for Z_t by solving the above ODEs.
(Hint: The solutions of an ODE $f'(x) = f(x)\gamma - 1$ are $f(x) = \frac{1}{\gamma} - Ce^{\gamma x}$ where *C* is a constant.)

Given that $\sigma = 2\%$, $r_0 = 3\%$, and the following information:

- Observed zero-coupon bond prices in the market are $Z_0(1\ year) = 0.97$, $Z_0(2\ year) = 0.94$, $Z_0(3\ year) = 0.87$, $Z_0(4\ year) = 0.83$, and $Z_0(5\ year) = 0.79$.
- Average yield of 1-month T-bill over the past 50 years is 5.1%.

Three potential choices for parameters of γ and $\bar{r}$ are given below:

	Choice 1	Choice 2	Choice 3
γ	0.9	0.45	0.45
$\bar{r}$	5%	5%	6%

(b) (*2 points*) Choose the best parametrization γ and $\bar{r}$. Show your work to support the choice.

The response for this part is to be provided in the Excel spreadsheet.

6. **Continued**

For the remaining parts, assume that interest rates follow the Vasicek model, and use a notional of 10,000 for all zero-coupon bonds.

You are given that $\gamma = 0.5$ and $\bar{r} = 5.5\%$, and the rest of the information is the same as above. Also, you are given that at $t = 0.5$, $r_{0.5} = 2.82\%$.

(c) (*1 point*) Determine the replicating portfolio at time 0, for the zero-coupon bond $Z_0(2\ year)$ using the zero-coupon bond $Z_0(4\ year)$ and cash.

The response for this part is to be provided in the Excel spreadsheet.

Assume that no rebalancing has been done until t=0.5.

(d) (*1 point*) Calculate, $P_{0.5}$, the value of the rebalanced replicating portfolio immediately after the rebalancing at time 0.5.

The response for this part is to be provided in the Excel spreadsheet.

(e) (*1 point*)

(i) Illustrate a relative value trade strategy using $Z_0(2\ year)$ and $Z_0(4\ year)$.

(ii) Calculate the profit of the strategy at time 0.5.

(f) (*1 point*) List two considerations when executing a relative value trade strategy as in part (e)(i).

7.

(*6 points*) As an option trader, you look at how to make profit from volatility arbitrage. To manage your risk, you use the Black-Scholes based Delta hedge and rebalance your hedges daily.

For the volatility used for hedging, you consider one of the following two choices:

- Implied volatility
- Actual volatility

(a) (*2 points*) List the pros and the cons of hedging with implied volatility and actual volatility.

(b) (*1 point*) Choose the most appropriate volatility for hedging under each of the following two constraints.

(i) Mark to model

(ii) Mark to market

Assume the following:

- The risk-free interest rate is 0.
- The spot price of a non-dividend-paying stock XYZ is 100.
- While the implied volatility σ(implied) of the at-the-money 1-year call option on XYZ is 20%, you predict that the actual volatility σ(actual) of XYZ for the next year will be 30%.

(c) (*2 points*) Design a volatility arbitrage to make money assuming that your prediction is correct and that you hedge with actual volatility.

(d) (*1 point*) Calculate the final profit from the arbitrage executed in part (c).

The response for this part is to be provided in the Excel spreadsheet.

8.

(*9 points*) For a non-dividend-paying stock XYZ, you consider taking two option strategies:

- Option Strategy A: a purchase of a one-year call option with a strike at 100 and simultaneously a sale of a one-year call option with a strike at 120 on a stock
- Option Strategy B: a purchase of a one-year call with a strike at 110 and simultaneously a sale of a one-year put with a strike at 90

Also assume that:

- The spot price of stock XYZ is 100
- The risk-free interest rate is 0%
- Implied volatility is constant 15%

For Option Strategy A:

(a) (*1 point*) Describe the key characteristics of its Delta profile.

(b) (*2 points*) Draw the Delta profile in the Excel spreadsheet.

The response for this part is to be provided in the Excel spreadsheet.

(c) (*1 point*) Describe the key characteristics of its Gamma profile.

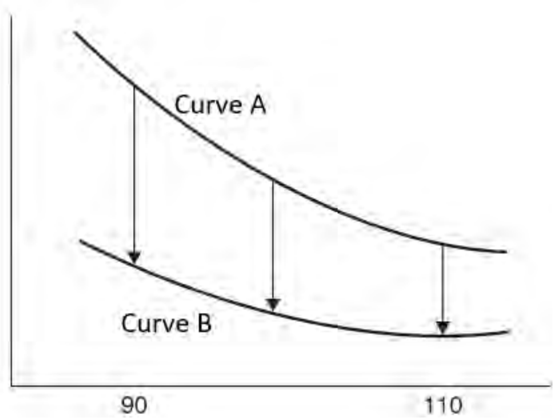
(d) (*1 point*) Describe their key characteristics of its Vega and Theta profiles.

8. Continued

For Option Strategy B:

(e) (*1 point*) Sketch the Vega profile and describe the key characteristics.

Assume that the volatility skew begins with *Curve A* with a negative slope, and then flattens to *Curve B*, as shown in the following chart.



(f) (*1 point*) Evaluate the impact of the change to Option Strategy B.

Now you consider using a Vanna-Volga approach to adjust the option price for Strategy B.

(g) (*1 point*) Define its *Vanna ratio* and *Vanna contribution*.

(h) (*1 point*) Describe how to apply the Vanna adjustment.

The responses for all parts of this question are required on the paper provided to you.

9.

(*7 points*) The Vanna Hull White Company (VHW) is a financial services company that specializes in boutique financial options. An agent from the VHW has approached your company to purchase $1,000,000 of a 1-year guarantee structured product featuring participation in the geometric average of a major stock index as described below.

$$\text{P} = \max\left(I, I * \left(1 + p_{dpa}\left(e^{\frac{1}{T}\int_0^T \ln\left(\frac{S_t}{S_0}\right)dt} - 1\right)\right)\right)$$

where I = initial investment amount, p_{dpa}= participation rate

Assume the current risk-free interest rate r = 3%, the current price of the underlying index (S_0) = 100, the current implied volatility (σ) = 20%, and the dividend rate q = 0%.

You are given that the price of a geometric mean Asian call option is equivalent to a vanilla European call option with a volatility of $\sigma_a = \frac{\sigma}{\sqrt{3}}$ and a dividend rate of $\frac{1}{2}\left(r + \frac{\sigma_a^2}{2}\right)$.

The time-*t* price of a European call option (for Black-Scholes model) is

$$N(d_1)\, S_t e^{-q(T-t)} - N(d_2)\, K * e^{-r(T-t)}$$

where $d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$,

S_t = index price at time *t*, *K* = strike price, *T* = maturity of the option.

The company is considering purchasing at-the-money (ATM) options and zero-coupon bonds (ZCBs) to hedge the 1-year guarantee structured product.

9. Continued

(a) (*2 points*)

(i) Identify the type of options which should be purchased.

(ii) Calculate the values in the table below, (assuming a Black-Scholes framework):

	Value
Risk Budget	
Number of ZCBs to purchase (1,000 Face value amount)	
Number of ATM options to purchase	
Fair participation rate	

A fellow actuary has expressed concern over how the options may perform if market volatilities increase significantly and has asked you to look into options to reduce the volatility risk.

(b) (*3 points*)

(i) (*2 points*) Determine the Vega of the Asian call options above.

(ii) (*1 point*) Explain the value of the above Vega in relation to the Vega of a European call option and why this relation intuitively makes sense.

The company's CRO has approached you to determine a Delta-Vega hedge strategy for the Asian options.

(c) (*2 points*) Determine an initial Delta-Vega hedge position using an ATM 1-year European call option and the underlying stocks.

The responses for all parts of this question are required on the paper provided to you.

10.

(*7 points*) For a variable annuity (VA) contract with a GMDB rider, the following are defined:

- The underlying asset for the fund is S_t, whose risk neutral SDE is $dS_t = rS_t dt + \sigma S_t dW_t$
  where r is the constant risk-free rate and σ is the constant volatility of S_t.
- F_t is value of the fund at time t. A fee of m is charged continuously on the fund value while the contract is in-force and thus $F_t = S_t e^{-mt}$.
- Before maturity of the contract at time T, the maximum of F_t and a guaranteed amount $G = S_0$ is paid upon death of the insured.
- Probability of the insured (contract issued at age x) surviving until time t is ${}_tp_x$, and the constant force of mortality is $\mu_{x+t} = \frac{d\ {}_tp_x/dt}{{}_tp_x}$.
- Mortality probability is independent of the fund return distribution.
- A zero-coupon bond with notional of 1 and maturity T has price $P_t = e^{-r(T-t)}$ at time t.
- The net liability L_t is value of the VA contract in excess of the account balance less the value of prospective fee income. Specifically, it can be expressed formulaically as

$$L_t = {}_tp_x(\Omega_t - \Upsilon_t) - {}_tp_x E^{\mathbb{Q}}\left[\int_t^T mF_s e^{-r(s-t)}{}_{s-t}p_{x+t} ds\right]$$

$$\Omega_t = E^{\mathbb{Q}}\left[\int_t^T e^{-r(T-t)} \max(G, F_s)_{s-t}p_{x+t}\mu_{x+s} ds\right]$$

$$\Upsilon_t = E^{\mathbb{Q}}\left[\int_t^T F_s e^{-r(T-t)}{}_{s-t}p_{x+t}\mu_{x+s} ds\right]$$

10. Continued

(a) (*2.5 points*) Derive the no-arbitrage value of the net liability L_t at time *t*.

Given that the Delta and Rho of L_t are as follow:

$$Delta = \frac{\partial L}{\partial S} = \int_t^T G e^{-m(s-t)}[N(d_1) - 1] \ {}_s p_x \mu_{x+s} ds$$
$$- m \int_t^T e^{-ms} \ {}_s p_x ds$$
$$Rho = \partial L \Big/ \partial r = -\int_t^T G(s-t) e^{-r(s-t)} N(-d_2) \ {}_s p_x \mu_{x+s} ds$$

where $d_1 = \frac{ln\frac{S_t}{G} + (s-t)(r - m + \frac{\sigma^2}{2})}{\sigma\sqrt{s-t}}$ and $d_2 = d_1 - \sigma\sqrt{s-t}$

(b) (*2 points*) Derive the positions of stock, zero-coupon bond and money market account for a portfolio Π_t that hedges the Delta and Rho of the net liability in part (a).

Assume that the hedging portfolio Π_t was developed based on the SDE $dS_t = r_t S_t dt + \sigma S_t dW_t$, where r_t was deterministic but time-dependent and calibrated to observed term structure.

The following steps are done to test the effectiveness of the hedging portfolio Π_t:

- Real world scenarios of S_t are simulated based on the SDE $dS_t = \mu S_t dt + \sigma S_t dW_t$.
- The interest rates are simulated stochastically using the two interest models below. Moreover, as a control, the interest rate is not simulated and assumed to follow the deterministic path r_t.
- Assume continuous re-balancing of the hedging portfolio.
- Effectiveness is assessed as the hedging gain/loss at time T (i.e., $\Pi_T - L_T$).

Control	Interest rate model 1	Interest rate model 2
Deterministic interest rate r_t	One-factor CIR model	Three-factor CIR model

(c) (*1.5 points*) Describe the hedging effectiveness you expect to observe under each of the 3 models of simulating interest rates (specified in the table above). Explain your reasoning.

10. Continued

Assume that the hedging portfolio Π_t was developed based on a short rate r_t that follows the one-factor Vasicek model calibrated to the observed term structure.

(d) (*1 point*) Describe changes in hedging effectiveness in comparison to part (c) for the Interest rate model 1 and the Interest rate model 2.

****END OF EXAMINATION****