EDUCATION COMMITTEE

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SOCIETY OF ACTUARIES

SHORT-TERM ACTUARIAL MATHEMATICS STUDY NOTE

SUPPLEMENT TO CHAPTER 3 OF

INTRODUCTION TO RATEMAKING AND LOSS RESERVING FOR PROPERTY AND CASUALTY INSURANCE, FOURTH EDITION

by

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CHAPTER 3 SUPPLEMENT (MARCH 5, 2018 UPDATE)

The methods presented in this chapter demonstrate how ultimate losses and reserves are calculated in practice from historical data. This section will focus on the statistical methods that underlie the methods presented in section 3.6.

We can restate Table 3.2 using the notation where $L_{i,k}$ denotes the cumulative loss payments for accident year i and development year k.

Table 3.19

Cumulative Loss Payments through Development Years								
	Development Year (*k*)							
Accident Year (i)	0	1	2	3	4	5	6	7
1	$L_{1,0}$	$L_{1,1}$	$L_{1,2}$	$L_{1,3}$	$L_{1,4}$	$L_{1,5}$	$L_{1,6}$	$L_{1,7}$
2	$L_{2,0}$	$L_{2,1}$	$L_{2,2}$	$L_{2,3}$	$L_{2,4}$	$L_{2,5}$	$L_{2,6}$	$L_{2,7}$
3	$L_{3,0}$	$L_{3,1}$	$L_{3,2}$	$L_{3,3}$	$L_{3,4}$	$L_{3,5}$	$L_{3,6}$	$L_{3,7}$
4	$L_{4,0}$	$L_{4,1}$	$L_{4,2}$	$L_{4,3}$	$L_{4,4}$	$L_{4,5}$	$L_{4,6}$	$L_{4,7}$
5	$L_{5,0}$	$L_{5,1}$	$L_{5,2}$	$L_{5,3}$	$L_{5,4}$	$L_{5,5}$	$L_{5,6}$	$L_{5,7}$
6	$L_{6,0}$	$L_{6,1}$	$L_{6,2}$	$L_{6,3}$	$L_{6,4}$	$L_{6,5}$	$L_{6,6}$	$L_{6,7}$
7	$L_{7,0}$	$L_{7,1}$	$L_{7,2}$	$L_{7,3}$	$L_{7,4}$	$L_{7,5}$	$L_{7,6}$	$L_{7,7}$
8	$L_{8,0}$	$L_{8,1}$	$L_{8,2}$	$L_{8,3}$	$L_{8,4}$	$L_{8,5}$	$L_{8,6}$	$L_{8,7}$

The values of $L_{i,k}$ for $i+k \leq K+1$ are known, where K is the highest development year, or 7 in this example. The values of $L_{i,k}$ for $i+k > K+1$ are unknown and represent the quantities that we want to estimate. These values are the run-off triangle, or future cumulative paid losses (the shaded values). The ultimate losses for each accident year are the values of $L_{i,K}$, or the last column of Table 3.19.

Table 3.19 uses loss payments, so the loss reserve (unpaid losses) for each accident year is represented by the following formula:

$$R_i = L_{i,K} - L_{i,K-i+1} \tag{3.9}$$

Formula 3.7a provided a formula for estimating ultimate losses, as

$$\textit{Estimated Ultimate Losses} = (\textit{Losses Paid-to-Date}) \cdot \prod_j f_j \qquad (3.7a)$$

where f_j is the loss-development factor from a paid-loss-development triangle at duration j (i.e., from development year $j - 1$ to j). We can restate formula 3.7a using the notation of this section, as:

$$L_{i,K} = (L_{i,K-i+1}) \cdot \prod_{j=K-i+2}^{K} f_j \qquad (3.10)$$

The age-to-age development factors can be calculated by several different formulas. Using the mean, or volume-weighted average, one such formula for f_j is:

$$f_j = \frac{\sum_{i=1}^{K-j+1} L_{i,j}}{\sum_{i=1}^{K-j+1} L_{i,j-1}} \qquad (3.11)$$

This formula can be obtained by making certain assumptions and then employing a statistical estimation technique. One such approach was developed by Mack (Mack, T., "Measuring the Variability of Chain Ladder Reserve Estimates," Casualty Actuarial Society Forum, Spring 1994, 101-182). Mack treats the L values as random variables with the following three properties:

1. $E(L_{i,k} \mid L_{i,0}, L_{i,1}, \ldots, L_{i,k-1}) = L_{i,k-1} f_k$
2. $Var(L_{i,k} \mid L_{i,0}, L_{i,1}, \ldots, L_{i,k-1}) = L_{i,k-1} \alpha_k^2$
3. $(L_{i,0}, L_{i,1}, \ldots, L_{i,K})$ and $(L_{j,0}, L_{j,1}, \ldots, L_{j,K})$ are independent for all $i \neq j$.

Here α_k^2 is a parameter that relates the variance to previous values in the same way f_k relates the mean to previous values. Properties 1 and 2 have two consequences. One is that the same factor applies regardless of the accident year (but does depend on the development year). The second is that a given value depends only on the previous development year and not on any prior years. For example, if unusually low development is observed between $L_{6,1}$ and $L_{6,2}$, the same development factor of f_3 is used regardless, thereby potentially understating the ultimate losses and consequently the loss reserve for accident year 6.

Mack does not make an assumption regarding the distribution of the L values. Hence maximum likelihood estimation cannot be used. Instead, he notes that given observations through development year j – 1, Property 1 implies that $L_{i,j} / L_{i,j-1}$ is an unbiased estimator of f_j. Then, for any set of weights, $w_{1,j}, \ldots, w_{K-j+1,j}$ with $w_{1,j} + \ldots + w_{K-j+1,j} = 1$,

$$\hat{f}_j = \sum_{i=1}^{K-j+1} w_{i,j} \frac{L_{i,j}}{L_{i,j-1}}$$

is also unbiased. The variance of a weighted average of independent estimators (implied by Property 3) is minimized when the weights are inversely proportional to the variance of each term. (See Example 3.5 at the end of this section for a proof when there are two estimators.) Using Property 2, given observations through development year j – 1,

$$\begin{aligned} w_{i,j} &\propto \frac{1}{Var(L_{i,j} / L_{i,j-1} \mid L_{i,0}, \ldots, L_{i,j-1})} \\ &= \frac{L_{i,j-1}^2}{Var(L_{i,j} \mid L_{i,0}, \ldots, L_{i,j-1})} \\ &= \frac{L_{i,j-1}^2}{L_{i,j-1}\alpha_j^2} = \frac{L_{i,j-1}}{\alpha_j^2}. \end{aligned}$$

The sum of these weights is

$$\sum_{i=1}^{K-j+1} \frac{L_{i,j-1}}{\alpha_j^2}.$$

To ensure the weights add to one, divide by this sum to obtain

$$w_{i,j} = \frac{\dfrac{L_{i,j-1}}{\alpha_j^2}}{\displaystyle\sum_{i=1}^{K-j+1} \frac{L_{i,j-1}}{\alpha_j^2}} = \frac{L_{i,j-1}}{\displaystyle\sum_{i=1}^{K-j+1} L_{i,j-1}}.$$

Finally,

$$\hat{f}_j = \sum_{i=1}^{K-j+1} \left(\frac{L_{i,j-1}}{\sum_{h=1}^{K-j+1} L_{h,j-1}} \right) \frac{L_{i,j}}{L_{i,j-1}} = \frac{\sum_{i=1}^{K-j+1} L_{i,j}}{\sum_{i=1}^{K-j+1} L_{i,j-1}}.$$

Mack shows that changing Property 2 regarding the variance leads to alternative formulas.

This stochastic form of determining the values of $L_{i,k}$ for $i+k > K+1$, or future cumulative paid losses in the run-off triangle, can be used to determine variances and therefore confidence intervals around the estimates of ultimate losses and consequently reserves.

We can also view the f_j values a different way to help formulate an alternative statistical representation of estimating ultimate losses. From formula 3.10, we can define:

$$f_{j,ult} = \prod_{h=j+1}^{K} f_h, \; j = 0, \ldots, K-1 \tag{3.12}$$

Formula 3.12 gives us the age-to-ultimate development factors for each accident year. We know from the Bornhuetter Ferguson method that $\left(1 - \frac{1}{f_{j,ult}}\right)$ gives us the ratio of ultimate losses yet to be paid after development period j to ultimate losses, therefore the ratio of ultimate losses paid by each development year j to ultimate losses is represented by:

$$r_j = \frac{1}{f_{j,ult}}, \; j = 0, \ldots, K-1 \tag{3.13}$$

We can then represent any value in the loss development triangle with the following relationship:

$$L_{i,j} = r_j \cdot L_{i,K} + e_{i,j}, \; j = 0, \ldots, K-1 \tag{3.14}$$

where $e_{i,j}$ is an error term. This formulation has the advantage that future values are not based solely on the most recent observed paid losses for each accident year. In addition, an error term can be incorporated directly into the formula. Formula 3.14 can then be used to model the future losses in the run-off triangle and also determine variances and therefore confidence intervals around the estimates of ultimate losses and consequently reserves.

The example in this section uses historical cumulative paid losses to develop future cumulative paid losses, including the ultimate losses. The same approach can be used with incurred losses.

Example 3.5

Let X and Y be two independent random variables with variances σ_X^2 and σ_Y^2, respectively. Let $Z = wX + (1 - w)Y$ be a weighted average of the two variables. Show that the variance of Z is minimized when $w = c/\sigma_X^2$ and $1 - w = c/\sigma_Y^2$ and determine the value of c.

Solution:

$$Var(Z) = w^2\sigma_X^2 + (1-w)^2\sigma_Y^2$$

$$\frac{\partial Var(Z)}{\partial w} = 2w\sigma_X^2 - 2(1-w)\sigma_Y^2 = 0$$

$$w = \frac{\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2} = \frac{1}{\sigma_X^2}\frac{\sigma_X^2\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2} = \frac{c}{\sigma_X^2}$$

$$1 - w = \frac{\sigma_X^2}{\sigma_X^2 + \sigma_Y^2} = \frac{1}{\sigma_Y^2}\frac{\sigma_X^2\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2} = \frac{c}{\sigma_Y^2}$$

where $c = \dfrac{\sigma_X^2\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}$.