Valuation of Equity-Indexed Annuities under Stochastic Interest Rates

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Abstract

This paper considers the pricing of equity-indexed annuities (EIAs). Traditionally, the values of the guarantees embedded in these contracts are priced by modeling the underlying index fund while keeping the interest rates constant. The assumption of constant interest rates becomes unrealistic in pricing and hedging the EIAs since the embedded guarantees are often of much longer maturity. To solve this problem, the authors propose an economic model that has the flexibility of modeling the underlying index fund as well as the interest rates. Some popular EIAs are illustrated to assess the implication of the proposed model.

1. Introduction

An equity-indexed annuity (EIA) is a fixed annuity that earns a minimum rate of interest and offers a potential gain that is tied to the performance of an equity index. The first EIA product was introduced in 1995 by Keyport Life Insurance Co. Since then, EIAs have enjoyed some popularity in both the United States and Canada. The sales of EIAs increased from \$1.5 billion in 1996 to \$5.2 billion in 1999. For years 2000 and 2001, the sales of EIAs remained stable at \$6 billion per year. Despite the downturn of the stock market, there seems to have been a renewed public interest in EIAs recently, as evidenced by the record-high sales of near \$12 billion in 2002. See the Advantage Group reports (Marrion 2000, 2001, 2002) for details.

EIAs appeal to investors because they not only offer some of the benefits underlying conventional annuities, but also offer participation in the equity market while limiting the downside risk. A typical EIA guarantees a minimum return (normally 3%) on a portion of the initial amount invested. In addition to this minimum guarantee, the annuitant receives some participation in the appreciation of a predetermined equity index such as the Standard & Poor's (S&P) 500. The indexing feature extends over a fixed term, typically ranging from one to 10 years.

There are several indexing methods for EIAs. In order of decreasing sales volumes, they are: annual reset, annual yield spread, point-to-point, high water mark, and term yield spread. The index growth on an EIA with annual reset option is measured each year by comparing the index level at the beginning and the end of the year. As in the annual reset method, the annual yield spread method resets the index growth annually, but a yield spread is deducted from the equity index. The index growth with point-to-point indexing is based on the growth between two time points. The index growth with a high water mark feature (also called term high point or point-to-point with discrete lookback) is calculated to the highest index anniversary value over the entire term of the annuity. Finally, the term yield spread method is similar to the annual yield spread method except that a yield spread is deducted for the entire term of the EIA. In addition to these methods, an averaging scheme is often used to calculate the index growth in order to reduce the costs of the guarantees and to be partly immunized from the market volatility.

There has been some research on this subject. See the monograph by Hardy (2003) (and also Tiong 2000) for comprehensive discussion on this

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topic. In general, it is often assumed that the equity index and interest rates are within a Black-Scholes framework; that is, the equity index follows a lognormal process with constant interest rates. In this paper, we consider a more general economic model, assuming that the interest rates are stochastic and follow a diffusion process. Our objective is to conduct a detailed numerical analysis on EIAs currently sold in the North American market and to examine the impact of the equity index and interest rates on the evaluation of these EIAs.

The paper is organized as follows. We present the joint economic model and valuation formulas in Sections 2, 3, and 4. In Section 5, we examine the implications of the proposed model on the conventional model by conducting a detailed analysis on the most popular types of embedded equity guarantees. Section 6 concludes the paper.

2. Economic Model Selections

Two crucial economic factors in the valuation and hedging of an EIA are the term structure of interest rates and the level of the equity index. The research to date has primarily focused on modeling just one of the key variables. For example, in the Black-Scholes framework, the index is stochastic while other variables, such as volatility or interest rates, are constant. While these assumptions might be adequate for most options offered by the exchanges and banks, it is sometimes undesirable to extrapolate that these assumptions are also applicable to the guarantees embedded in EIAs. Most of the options offered by the exchanges and banks typically are short-dated with maturity less than one year, and, hence, a Black-Scholes framework would provide a reasonable approximation for pricing purposes. In contrast, the embedded guarantees associated with EIAs have maturities ranging from one to 10 years. It is, therefore, unreasonable to assume that the interest rates would remain level for such a long duration.

In this section we jointly model the term structure of interest rates and the equity index using stochastic differential equations. The basic model consists of a stochastic differential equation for the short-term interest rate and a stochastic differential equation for the equity index. We wish that (1) the short rate process reproduces the current yield curve as well as bond prices at different maturities; (2) the model incorporates correlation between the interest rate and the equity index, which is accomplished by explicitly introducing correlation between the diffusion processes; and (3) the model is mathematically or computationally tractable in the sense it can be implemented using sophisticated mathematical or numerical tools such as the Monte Carlo simulation.

We now describe the interest rate and equity processes. Let $r(t)$ be the short rate at time t for $0 \leq t \leq \mathcal{T}$, where $\mathcal{T}$ is the time-to-maturity of an EIA. We assume that the short rate process $\{r(t)\}$, $0 \leq t \leq \mathcal{T}$, satisfies the following stochastic differential equation:

$$dr(t) = \mu_r(t, r)\, dt + \sigma_r(t, r)\, dW_r(t), \qquad (2.1)$$

where the drift $\mu_r(t, r) = \kappa[\theta(t) - r]$ for some deterministic function $\theta(t)$, the volatility of the short rate process $\sigma_r(t, r) = \sigma_r$, and $\{W_r(t)\}$ is a standard Brownian motion. Let $S(t)$ be the equity index level at time t, which is governed by a stochastic differential equation of the form:

$$dS(t) = \mu_S(t)S(t)\, dt + \sigma_S(t)S(t)\, dW_S(t). \quad (2.2)$$

Here $\mu_S(t)$ represents the instantaneous rate of return of the index at time t and $\sigma_S(t) > 0$ is the volatility of the index at time t, both of which are deterministic, and $\{W_S(t)\}$ is a standard Brownian motion that is correlated with $\{W_r(t)\}$ with correlation coefficient ρ; that is,

$$corr(W_r(t), W_S(t)) = \rho.$$

For valuation purposes, the underlying probability measure associated with these processes is assumed to be the risk-neutral measure or Q-measure under which the present value process

$$V(t) = e^{-\int_0^t r(u)\, du}\, S(t) \qquad (2.3)$$

is a martingale.[1] Hence we have $\mu_S(t) = r(t)$.

We now make some remarks. The short rate model (2.1) is often referred to as the extended Vasicek model (Vasicek 1977) or the Hull and White model (Hull and White 1990). It can be

[1] A stochastic process $\{V(t)\}$ is a martingale if, for any $t < s$, $E[V(s)|\mathcal{F}_t] = V(t)$. See Øksendal (1998, p. 31).

shown that a closed-form solution for $r(t)$ exists and is given by

$$r(t) = r(0)e^{-\kappa t} + \kappa \int_0^t e^{-\kappa(t-u)}\theta(u)\, du + \sigma_r \int_0^t e^{-\kappa(t-u)}\, dW_r(u). \quad (2.4)$$

As a result, the price of default-free discount bonds with all possible maturities may be derived explicitly, since, for a discount bond maturing at time t, its price is given by $\mathrm{E}(e^{-\int_0^t r(s)\,ds})$. This, thus, allows the current zero-rate curve to be derived. Hence, we could choose a proper functional form of $\theta(t)$ such that the model zero-rate curve coincides with the market zero-rate curve. However, in this paper, it is assumed that $\theta(t)$ remains a constant; that is, $\theta(t) = \theta$ for all t, for illustration purposes, as considered in Chan et al. (1992), Ait-Sahalia (1996), and Duan and Simonato (1999). Implementation of the Model (2.1) with a deterministic function can be accommodated with minor modifications. As we start with the risk-neutral probability measure for the model, the parameter θ is of the form $\theta = \theta_0 + \sigma\lambda/\kappa$, where θ_0 represents the long-term average of the short rate $r(t)$ and λ is the market price of risk under the physical measure. It should be pointed out that the model could generate negative short rates since $r(t)$ is a normal random variable for each t. However, in most practical applications, the probability of having a negative interest rate is very small and, hence, it still serves as a reasonable model due to its tractability. For further details on term structure interest rate models and their parameter estimation issues, see James and Webber (2000).

Let $C(s)$ be the payoff in year s for an initial one monetary unit investment in an EIA. By definition, the payoff $C(s)$ is a contingent claim, as it depends on the values of the index and the short rates up to time s. Thus, in the absence of mortality risk, the time-t price of $C(s)$, $t < s$, is given by

$$\Pi(t, s) = \mathrm{E}[e^{-\int_t^s r(u)\,du}\, C(s) | \mathcal{F}_t], \quad (2.5)$$

where $\mathcal{F}_t$ is the time-t information structure generated by $\{r(t)\}$ and $\{S(t)\}$ (Harrison and Pliska 1981). Intuitively, the time-t price is simply the expected (short rate) discounted payoff under the risk-neutral probability measure Q. In a special case where $t = 0$, the time-0 price becomes

$$\Pi(0, s) = \mathrm{E}[e^{-\int_0^s r(u)\,du}\, C(s)].$$

It is easy to see that using the Law of Iterated Expectations, the previous expression is equivalent to

$$\Pi(0, s) = \mathrm{E}[e^{-\int_0^t r(u)\,du}\, \Pi(t, s)].$$

Furthermore, since $\mathrm{E}[e^{-\int_0^s r(u)\,du} C(s) | \mathcal{F}_t]$ is a martingale, $e^{-\int_0^t r(u)\,du}\Pi(t,s)$ is also a martingale. It follows from $\sigma_S(t) > 0$ and the Martingale Representation Theorem (see, e.g., Øksendal 1998, p. 53) that

$$d[e^{-\int_0^t r(u)\,du}\, \Pi(t, s)] = \delta(t, s) dV(t), \quad (2.6)$$

for a (predictable) stochastic process $\delta(t, s)$. Here, the present value of the index, $V(t)$, is given in Equation (2.3). Applying the multiplication rule of the Itô calculus, we obtain

$$d\Pi(t, s) = r(t)\phi(t, s)\, dt + \delta(t, s)\, dS(t), \quad (2.7)$$

with

$$\phi(t, s) = \Pi(t, s) - \delta(t, s)S(t). \quad (2.8)$$

Thus, we may replicate the payoff $C(s)$ using a self-financing trading strategy $\{\phi(t, s), \delta(t, s)\}$, where the replicating portfolio consists of the amount $\phi(t, s)$ in the money market account and $\delta(t, s)$ units of the index fund at time t.

3. INCORPORATING MORTALITY RISK

In this section, we incorporate mortality risk into our analysis. We will discuss how to evaluate an EIA using the economic model in Section 2 while taking into account the mortality risk. As we saw in the last section, a fundamental idea of pricing a contingent payoff in the financial market using the risk-neutral probability measure is that one can perfectly replicate the payoff of the contingent claim by rebalancing a portfolio consisting of the underlying risky asset(s) and the money market account with a self-financing strategy.

A necessary condition for this pricing principle to hold is that all financial assets involved must be tradable over time. When a payoff not only depends on risky assets but also is contingent on the mortality of the holder, this condition is violated and the pricing principle discussed in the last section is no longer applicable. Consequently, a perfect hedging using a self-financing trading strategy becomes impossible. A possible solution to overcome this is to consider the risk-minimizing hedging strategy proposed by Föllmer and Sondermann (1986) (also see Møller 1998 and 2001a, b). We now describe the underlying principle associated with this method.

We begin by introducing necessary actuarial symbols. As in Bowers et al. (1997), let (x) be an annuitant who purchases an EIA at age x and $T(x)$ be the future lifetime of (x). Also let ${}_tp_x$ and ${}_tq_x$ be the probability of survival and death, respectively; that is, ${}_tp_x = P(T(x) > t)$ and ${}_tq_x = 1 - {}_tp_x$, with the convention that ${}_1q_x = q_x$. Furthermore, the force of mortality is defined as $\mu(x + t) = -\frac{d}{dt} \ln {}_tp_x$.

We assume that the future lifetime $T(x)$ is stochastically independent of the Brownian motions $W_r(t)$ and $W_S(t)$. Consequently, $T(x)$ is also stochastically independent of $r(t)$ and $S(t)$. Intuitively, this means that the event of death is independent of the interest rates and the equity index, and vice versa; hence, the mortality risk can be diversified by selling policies of the same kind to enough people.[2]

Let $C(t)$ be the payoff at time t of an EIA. For simplicity, we assume that there is no surrender and death benefits are paid at the end of the year of death. The present value of the payoff, C, can be defined as

$$C = \sum_{j=0}^{\mathcal{T}-1} e^{-\int_0^{j+1} r(u)\,du}\, C(j+1) I(j < T(x) \le j+1) + e^{-\int_0^{\mathcal{T}} r(u)\,du}\, C(\mathcal{T}) I(T(x) > \mathcal{T}), \quad (3.1)$$

where $\mathcal{T}$ is the time of maturity and $I(A)$ is the indicator function of event A; that is, $I(A) = 1$ if A is true, $I(A) = 0$ otherwise. Note that, in practice, the final payoff $C(\mathcal{T})$ might not be of the same function form as that of the death benefits, but for simplicity we assume it to be the same.

Now consider a portfolio that consists of the index fund and a money market account that earns interest at rate $r(t)$ at time t. It is required that this portfolio will meet the liability obligation given above. Let $\delta(t)$ be the number of units of the index fund at time t and $\phi(t)$ be the amount in the money market account at time t. Thus, the value of this portfolio, $PO(t)$, at time t is given by

$$PO(t) = \phi(t) + \delta(t)S(t),$$

and its present value, $PV(t)$,

$$\begin{aligned} PV(t) &= e^{-\int_0^t r(u)\,du}\, PO(t) \\ &= e^{-\int_0^t r(u)\,du} \phi(t) + \delta(t)V(t). \end{aligned}$$

Note that we have the condition $PV(\mathcal{T}) = C$ to ensure meeting the liability obligation. The present value of the accumulated costs up to time t for maintaining the portfolio from the above trading strategy is, thus,

$$CO(t) = PV(t) - \int_0^t \delta(u)\, dV(u), \quad (3.2)$$

where $\int_0^t \delta(u)\, dV(u)$ represents the present value of the accumulated capital gains at time t. If the trading strategy $\{\phi(t), \delta(t)\}$ were self-financing, then it follows from Equation (2.6) that

$$d[CO(t)] = d[PV(t)] - \delta(t)\, dV(t) = 0.$$

In this case, $CO(t)$ would be a constant and $CO(t) = CO(0) = PV(0)$. Obviously, the trading strategy $\{\phi(t), \delta(t)\}$ under the present consideration is not self-financing due to the embedded mortality risk associated with the payoff $C(t)$. Hence, we seek a trading strategy

[2] We compromised the mathematical rigor in this discussion in order to present the intuitive idea of how to integrate the mortality risk into the model. Mathematically, we need to expand the information structure $\{\mathcal{F}_t\}$ associated with $r(t)$ and $S(t)$ and the risk neutral probability measure Q to include the mortality law. The new information structure $\{\tilde{\mathcal{F}}_t\}$ is such that, for each t, $\tilde{\mathcal{F}}_t$ is the smallest information structure containing $\mathcal{F}_t$ and the information structure generated by $T(x)$ while maintaining the independence between $\{\mathcal{F}_t\}$ and the information structure generated by $T(x)$. We denote this the new risk-neutral probability measure $\tilde{Q}$. It should be noted that there are many ways to expand the risk-neutral probability measure Q to all events in $\{\tilde{\mathcal{F}}_t\}$ and the one we adopt is called the "minimal martingale measure" by Schweizer (1991, 1995). Furthermore, the mortality law remains the same. For a detailed description, see Møller (1998, Sect. 2.3). For notational simplicity, we again denote $\{\tilde{\mathcal{F}}_t\}$ and $\tilde{Q}$ by $\{\mathcal{F}_t\}$ and Q, respectively. It should also be pointed out that the independence between investment risk and mortality risk in the physical measure does not necessarily imply their independence in the risk-neutral measure.

$\{\phi(t), \delta(t)\}$ that minimizes the variance of the cost process $\{CO(t)\}$ under the risk-neutral probability measure. A strategy of this kind is called the risk-minimizing trading strategy (see, e.g., Föllmer and Sondermann 1986, or Møller 1998). More precisely, the risk-minimizing trading strategy $\{\phi(t), \delta(t)\}$ is the optimal stochastic process for the following optimization problem:

$$\text{Minimize}_{\phi,\delta}\ \mathrm{E}[(CO(\mathcal{T}) - CO(t))^2], \quad \text{for all } 0 \le t \le \mathcal{T}. \quad (3.3)$$

It can be shown (see Møller 1998) that the optimal solutions of the optimization problems, $PV(t)$, $\delta(t)$ and $\phi(t)$ is given as follows:

- For $t = 0, 1, \ldots, \mathcal{T}$,

$$PV(t) = \mathrm{E}\{C|\mathcal{F}_t\} = \mathrm{E}\{\mathrm{E}[C|T(x)]|\mathcal{F}_t\}$$
$$= \sum_{j=0}^{t-1} e^{-\int_0^{j+1} r(u)\,du} C(j+1) I(j < T(x) \le j+1)$$
$$+ I(T(x) > t) e^{-\int_0^t r(u)\,du} \sum_{j=t}^{\mathcal{T}-1} \Pi(t, j+1)_{j-t}\, p_{x+t}\, q_{x+j}$$
$$+ I(T(x) > t) e^{-\int_0^t r(u)\,du} \Pi(t, \mathcal{T})_{\mathcal{T}-t} p_{x+t}, \quad (3.4)$$

where $\Pi(t, j)$ as defined in equation (2.5). If time t is between two payment dates, that is, if there is an integer k such that $k < t < k + 1$, then $PV(t)$ is obtained by (i) replacing t by k in the summation limits, (ii) adding $e^{-\int_0^t r(u)\,du}\Pi(t, k+1) I(k < T(x) \le t)$, and (iii) replacing q_{x+t} by ${}_{k-t+1}q_{x+t}$ in the first term of the second summation of equation (3.4). For the special case where $t = 0$, the previous expression simplifies to

$$PV(0) = \sum_{j=0}^{\mathcal{T}-1} \Pi(0, j+1)\,{}_j p_x\, q_{x+j} + \Pi(0, \mathcal{T})\,{}_{\mathcal{T}} p_x. \quad (3.5)$$

- For $t = 0, 1, \ldots, \mathcal{T}$,

$$\delta(t) = I(T(x) > t) \times \left[\sum_{j=t}^{\mathcal{T}-1} \delta(t, j+1)_{j-t}\, p_{x+t}\, q_{x+j} + \delta(t, \mathcal{T})_{\mathcal{T}-t} p_{x+t} \right], \quad (3.6)$$

where $\delta(t, j)$ as given in equation (2.6). Similarly, if $k < t < k + 1$, the above equation becomes

$$\delta(t) = I(k < T(x) \le t)\delta(t, k+1) + I(T(x) > t)\Bigg[\delta(t, k+1)_{k-t+1} q_{x+t} + \sum_{j=k+1}^{\mathcal{T}-1} \delta(t, j+1)_{j-t}\, p_{x+t} q_{x+j} + \delta(t, \mathcal{T})_{\mathcal{T}-t} p_{x+t} \Bigg]. \quad (3.7)$$

- Finally, $\phi(t) = e^{\int_0^t r(u)\,du} PV(t) - \delta(t) S(t)$.

To conclude this section, we remark that the above results can be generalized to benefits that are payable at the moment of death. In this case, expressions (3.4) and (3.6) become, respectively,

$$PV(t) = \int_0^t e^{-\int_0^s r(u)\,du} C(s)\, dI(T(x) \le s) + I(T(x) > t) e^{-\int_0^t r(u)\,du} \int_t^{\mathcal{T}} \Pi(t, s)_{s-t} p_{x+t}\, \mu(x+s)\, ds + I(T(x) > t) e^{-\int_0^t r(u)\,du} \Pi(t, \mathcal{T})_{\mathcal{T}-t} p_{x+t},$$

and

$$\delta(t) = I(T(x) > t) \times \left[\int_t^{\mathcal{T}} \delta(t, s)_{s-t} p_{x+t}\, \mu(x+s)\, ds + \delta(t, \mathcal{T})_{\mathcal{T}-t} p_{x+t} \right].$$

4. PRICING EIAS

In this section, we focus on pricing and reserving issues of EIAs using the framework presented in the previous section. As we discussed earlier, an EIA is priced through its participation in the index growth. For an EIA with an initial value of one monetary unit, the payoff $C(t)$ at time t and, hence, $\Pi(0, t)$, are both functions of the EIA's participation rate or the spread between the index return and the EIA's credited return. For example, a plain vanilla point-to-point EIA has the following payoff structure[3]:

$$C(t) = \max[\min[1 + \alpha R_t, (1 + \zeta)^t], \beta(1 + g)^t], \tag{4.1}$$

where R_t measures the "appreciation" of the index up to year t, the participation rate α determines the proportion of the growth of the index fund to be credited, the cap or ceiling rate ζ is the annualized maximum rate that can be credited, the parameter g is the minimum annualized guarantee rate (or the floor rate) for the entire term of the contract, and β is the percentage of unit for which the minimum guarantee is applied.

Consequently, the present value of the EIA (i.e., $PV(0)$, as given in equation 3.5) is also a function of the participation rate. As the initial value of the EIA is equal to 1, we have $PV(0) = 1$; and it follows from equation (3.5) that the participation rate α is a solution to the following pricing equation:

$$\sum_{j=0}^{\mathcal{T}-1} \Pi(0, j + 1)\,{}_jp_x\, q_{x+j} + \Pi(0, \mathcal{T})\,{}_{\mathcal{T}}p_x = 1. \tag{4.2}$$

We call the implied participation rate satisfying the above equation the *critical* or *fair* participation rate. By holding all other parameter values constant, the critical participation rate always exists because the expression on the left-hand side of equation (4.2) is an increasing function in α. Also, the pricing formula (4.2) implies that the calculation is based on a net premium basis, meaning that the risk-minimizing hedging strategy discussed in the last section is employed or the mortality risk is diversified and that other charges, such as the management and expenses charges, are not taken into account.

The same argument may be used to set reserves over time. Let $R(t)$, for $t = 0, 1, \cdots, \mathcal{T}$, be the reserve at time t. Under the risk-minimizing hedging strategy, we have

$$R(t) = e^{\int_0^t r(u)\,du} \mathrm{E}\{C | T(x) > t\}$$

so that this quantity can be computed as

$$R(t) = \sum_{j=t}^{\mathcal{T}-1} \Pi(t, j + 1)\,{}_{j-t}p_{x+t}\, q_{x+j} + \Pi(t, \mathcal{T})\,{}_{\mathcal{T}-t}p_{x+t}. \tag{4.3}$$

An obvious drawback of the above approach is that the mortality risk is not well addressed, particularly when dealing with a small number of policies. To partly overcome this problem, we may consider the use of the percentile premium principle described in Bowers et al. (1997, Chap. 6). Consider a portfolio of n independent and identical policies. Under this principle, we have the following alternative pricing formula:

$$\mathrm{E}(C) + (\varepsilon/\sqrt{n})\sqrt{\mathrm{Var}\{\mathrm{E}[C|T(x)]\}} = 1, \tag{4.4}$$

where C is the representative of the present value of any of these policies' payoff, and ε is, say, the 95th percentile of the distribution of the normalized random variable of $\mathrm{E}[C|T(x)]$. For a small value of n such as $n = 1$, the parameter ε may be estimated using the smoothed empirical estimation of ε (see Klugman, Panjer, and Willmot 1998, p. 35). For a relatively large n (say $n \geq 20$), a normal approximation of the distribution would be appropriate and we may let $\varepsilon = 1.96$. We refer the participation rate implied from equation (4.4) as the *loaded participation rate.* Intuitively, the loaded participation rate guarantees that, for each policy, there are sufficient funds covering EIA benefits 95% of the time. Consequently, a loaded participation rate from equation (4.4) is always lower than the critical participation rate from equation (4.2). However as n increases, the loaded participation rate approaches the critical participation rate.

The calculation of loaded participation rates does not require much additional effort relative to computing the critical participation rate that satisfies equation (4.2). This follows because

$$\mathrm{Var}\{\mathrm{E}[C|T(x)]\} = \mathrm{E}\{[\mathrm{E}[C|T(x)]]^2\} - [\mathrm{E}(C)]^2$$

where

$$\mathrm{E}\{[\mathrm{E}[C|T(x)]]^2\} = \sum_{j=0}^{\mathcal{T}-1} [\Pi(0, j + 1)]^2\,{}_jp_x\, q_{x+j} + [\Pi(0, \mathcal{T})]^2\,{}_{\mathcal{T}}p_x,$$

and the quantities $\Pi(0, s)$ and $\mathrm{E}(C) = PV(0)$ are already known from computing the critical par-

[3] This and other EIAs are discussed in detail in the next section.

ticipation rate. The corresponding loaded reserve is then obtained via

$$R_L(t) = e^{\int_0^t r(u)\,du}\Big\{ \mathrm{E}(C|T(x) > t) + (\varepsilon/\sqrt{n})\sqrt{\mathrm{Var}\{\mathrm{E}[C|T(x+t)]\}} \Big\}. \quad (4.5)$$

Both equations (4.2) and (4.4) are the basis for the numerical implementation in the next section. Note that the risk-neutral probability measure is used in the previous formulas as well as in the numerical implementation when calculating the conditional expectations, and that the mortality law is used to calculate the outside expectations (see footnote 2 for an explanation).

5. NUMERICAL IMPLEMENTATION

In this section, we analyze numerically the proposed model by considering several types of EIAs. Our examples involve seven-year EIAs issued to a life age 58 with a minimum interest rate guarantee of either 3% on 100% of the premium or 3% on 90% of the premium. In other words, we assume that the annuitant age 58 would hold the policy for seven years, contingent on survival, and convert it to a traditional fixed annuity at retirement age 65. The mortality of the annuitant is assumed to follow the 1979–1981 U.S. Life Table (see Bowers et al. 1997, Table 3.3.1).

The index fund under consideration is governed by a geometric Brownian motion with the initial value normalized to one unit. The volatility of the index is assumed to be constant and is either 20% or 30%. For the interest rate model (2.1), we use the same parameter values estimated by Ait-Sahalia (1996), that is, $\kappa = 0.85837$ and $\theta = 0.089102$. This implies that the long-term mean interest rate is around 8.9% and the mean-reverting intensity is 0.85837. We also assume the initial interest rate $r_0 = 0.05$, and the volatility of the interest rate takes the values of {0%, 4%, 8%} to reflect the current interest rate environment and to examine the impact of the volatility of the interest rate.

Note that, although zero volatility represents that the interest rate is deterministic, the rates need not be constant over time. Hence, our numerical results also provide a comparison between the valuation under deterministic interest rate and the valuation under stochastic interest rate. In the following subsections, we will first consider the critical participation rate (or the critical spread or cap as will be seen later) based on pricing formula (4.2) and the loaded participation rate under the alternative pricing formula (4.4).

5.1 Point-to-Point EIAs

Let us consider one of the simplest classes of EIAs. The crediting strategy for this family is known as the *point-to-point* design and, in general, its contingent claim $C(t)$ in year t for one unit of EIA can be represented as that in equation (4.1); that is,

$$C(t) = \max[\min[1 + \alpha R_t, (1 + \zeta)^t], \beta(1 + g)^t]. \quad (5.1)$$

The above payoff structure is ideal for many investors. While subject to the maximum cap rate that can be earned under this design, the first random term in equation (5.1) allows the investors to participate in any potential upside gain in the equity market. More importantly, in the event of an adverse market environment, the downside risk is constrained to the minimum guarantee floor component, that is, $\beta(1 + g)^t$. The presence of the cap rate, although placing an upper bound on the rate of return of the contract, could reduce the cost of such design substantially. In this subsection, we will first focus on identifying a critical participation rate, while fixing g, β and ζ. Various variations of point-to-point EIAs exist, depending on how we define the equity return random variable R_t in equation (5.1). As R_t denotes the "gain" for the reference index over the time interval $(0, t]$, without loss of generality, we can redefine R_t as

$$R_t = \frac{S^*(t)}{S(0)} - 1, \quad (5.2)$$

where $S^*(t)$ is an appropriate variable yet to be determined. Using this representation, the variable $C(t)$ can be formulated as the following two equivalent expressions:

$$C(t) = \beta(1 + g)^t + \frac{\alpha}{S(0)} \{\max[S^*(t) - K_g, 0] - \max[S^*(t) - K_\zeta, 0]\} \quad (5.3)$$

or

$$C(t) = (1 + \zeta)^t - \frac{\alpha}{S(0)} \times \{\max[K_\zeta - S^*(t), 0] - \max[K_g - S^*(t), 0]\} \quad (5.4)$$

where

$$K_\zeta = \frac{S(0)}{\alpha}[(1 + \zeta)^t - 1 + \alpha]$$

and

$$K_g = \frac{S(0)}{\alpha}[\beta(1 + g)^t - 1 + \alpha].$$

These expressions provide additional insight on hedging the EIA products. The first representation (5.3) suggests that one hedging approach is to construct a portfolio with the following three positions: (i) a zero-coupon bond that matures to $\beta(1 + g)^t$ in year t, (ii) a long position of $\alpha/S(0)$ units of call options with strike price K_g, and (iii) a short position of $\alpha/S(0)$ units of call options with strike price K_ζ. The hedging strategy involving (ii) and (iii) is often referred to as a "bull spread," a common practice in option trading (see Hull 2000, pp. 187–89). Alternatively, from equation (5.4), an equivalent strategy is to simultaneously invest in a t-year zero-coupon bond maturing to $(1 + \zeta)^t$ and a short bear spread (i.e., a long and a short position of $\alpha/S(0)$ units of put options with strike prices K_g and K_ζ, respectively).

Note that the above formulations can easily be adjusted for a design that does not place an upper bound on the rate of return. This is accomplished by setting $\zeta = \infty$ so that $\max[S^*(t) - K_\zeta, 0] = 0$. Then, equation (5.3) simplifies to

$$C(t) = \beta(1 + g)^t + \frac{\alpha}{S(0)} \max[S^*(t) - K_g, 0]. \quad (5.5)$$

Similarly, we have $\max[K_\zeta - S^*(t), 0] = K_\zeta - S^*(t)$ when $\zeta = \infty$, so that equation (5.4) reduces to

$$C(t) = 1 - \alpha + \frac{\alpha}{S(0)} \{S^*(t) + \max[K_g - S^*(t), 0]\}. \quad (5.6)$$

This analysis implies that the pricing of the EIAs with point-to-point design boils down to the valuation of the corresponding call (or put) options. This, in turn, depends on how the key variable $S^*(t)$ is determined. In practice, various forms of $S^*(t)$ have been proposed. In this paper, we consider three of the most popular structures, namely the long-term or *term-end point* design, the *Asian-end* design, and the *high water mark* or *term-high point* design. We now discuss these features in greater details.

The term-end point design, which is also the simplest crediting method, defines $S^*(t)$ as the observed index value at the end of the year of surrendering the contract; that is,

$$S^*(t) = S(t). \quad (5.7)$$

This particular case is easy to analyze since the cost of the EIA reduces to a portfolio of standard call (or put) options. Analytic solutions of these options are possible under certain interest rate models including the Vasicek model.[4] The results in Tables 1, 2, and 3 were obtained analytically.

Table 1 compares the critical and loaded participation rates implied from equations (4.2) and (4.4), respectively, over various sets of parameter values. For the loaded case, we consider $n = 20$ and 100. In all these cases, the EIAs are based on seven-year term endpoint crediting designs without the ceiling cap rate. We also consider two types of minimum guarantee: one with $g = 3\%$ and $\beta = 100\%$ and the other with $g = 3\%$ and $\beta = 90\%$. Based on these results, we make the following remarks:

REMARK 1A

As expected, the loaded participation rates are consistently lower than the corresponding unloaded critical participation rates.

REMARK 1B

The critical (or loaded) participation rates for cases with $\beta = 100\%$ are consistently lower than the corresponding values on 90% premium. With the lower proportion guarantee on the premium, the cost of EIAs must be lower. Thus, the participation rates must increase to raise the cost of the EIAs to unity.

[4] The pricing formulas for equity European call or put options with the Vasicek interest rate process are given in Rabinovitch (1989).

Table 1
Point-to-Point with Term-End Design (without Cap)

		Critical Participation Rates Using Equation (4.2)			Loaded Participation Rates Using Equation (4.4)					
					with n = 20			with n = 100		
		Correlation			Correlation			Correlation		
σ_S	σ_r	−30%	0	30%	−30%	0	30%	−30%	0	30%
3% Minimum Guarantee on 100% Premium										
20%	0%	85.95	85.95	85.95	84.88	84.88	84.88	85.49	85.49	85.49
	4	86.62	85.04	83.55	85.57	83.98	82.49	86.17	84.58	83.09
	8	85.42	82.25	79.45	84.39	81.21	78.42	84.98	81.80	79.00
30	0	73.32	73.32	73.32	72.03	72.03	72.03	72.76	72.76	72.76
	4	73.89	72.54	71.28	72.60	71.27	70.02	73.33	71.99	70.73
	8	72.86	70.16	67.79	71.60	68.94	66.60	72.31	69.63	67.28
3% Minimum Guarantee on 90% Premium										
20	0	90.89	90.89	90.89	90.40	90.40	90.40	90.68	90.68	90.68
	4	91.49	90.20	88.97	91.00	89.71	88.47	91.28	89.99	88.75
	8	90.67	88.05	85.66	90.21	87.57	85.18	90.47	87.84	85.45
30	0	79.94	79.94	79.94	79.21	79.21	79.21	79.62	79.62	79.62
	4	80.52	79.29	78.14	79.79	78.57	77.43	80.20	78.98	77.83
	8	79.80	77.32	75.12	79.10	76.64	74.46	79.50	77.03	74.83

REMARK 1C

As we increase the volatility of the index fund, the critical (or loaded) participation rate declines. This is to be expected because the more volatile the fund is, the greater the appreciation of the fund. Since the value of the EIA must remain at unity, the critical (or loaded) participation rate must necessarily be lower in order to compensate for the greater gain from the index. An alternate argument is to revisit the option decompositions in equation (5.5) or (5.6). For instance, let us consider equation (5.5) which involves the embedded call option. The value of a call option increases with the volatility of the underlying in-

Table 2
Point-to-Point with Term-End Design and 20% Cap Rate

		Critical Participation Rates Using Equation (4.2)			Loaded Participation Rates Using Equation (4.4)					
					with n = 20			with n = 100		
		Correlation			Correlation			Correlation		
σ_S	σ_r	−30%	0	30%	−30%	0	30%	−30%	0	30%
3% Minimum Guarantee on 100% Premium										
20%	0%	91.22	91.22	91.22	90.42	90.42	90.42	90.88	90.88	90.88
	4	91.03	90.48	90.06	90.25	89.68	89.23	90.70	90.13	89.71
	8	89.38	88.10	87.32	88.61	87.28	86.44	89.05	87.75	86.94
30	0	91.66	91.66	91.66	90.44	90.44	90.44	91.14	91.14	91.14
	4	90.17	90.69	91.33	89.01	89.47	90.04	89.67	90.16	90.77
	8	86.98	87.63	88.73	85.87	86.42	87.40	86.50	87.11	88.16
3% Minimum Guarantee on 90% Premium										
20	0	97.48	97.48	97.48	97.33	97.33	97.33	97.41	97.41	97.41
	4	97.00	97.07	97.25	96.86	96.92	97.10	96.93	97.00	97.18
	8	95.73	95.65	96.01	95.59	95.50	95.84	95.66	95.58	95.93
30	0	104.36	104.36	104.36	104.07	104.07	104.07	104.23	104.23	104.23
	4	102.21	103.64	105.21	101.93	103.35	104.90	102.08	103.50	105.07
	8	98.91	101.31	104.24	98.65	101.01	103.91	98.79	101.17	104.09

dex. Hence, to dampen the increased value of the call option, the strike price K_g is adjusted upward by reducing the participation rate α in order to satisfy equation (4.4) or (4.2). Also note that the participation rate is sensitive to the index volatility.

REMARK 1D

Holding all other parameters constant, the critical (or loaded) participation rate is a nonincreasing function in ρ. This follows from the property of the call option that its value is a nondecreasing function in ρ.

REMARK 1E

The results corresponding to $\sigma_r = 0$ represent an economy with deterministic interest rates. This is a generalization of the Black-Scholes framework, in that the implied yield curve from the Vasicek model with $\sigma_r = 0$ is not necessary flat. In fact, for the parameter values considered in our examples, the term structure of interest rates is upward sloping. These values, therefore, provide a benchmark when we induce randomness into the underlying interest rates.

REMARK 1F

For $\rho \geq 0$, the critical (or loaded) participation rate declines monotonically as the volatility of the interest rate increases. Alternatively, for the negatively correlated cases, the critical (or loaded) participation rates increase initially with the volatility of the interest rates. As the volatility of the interest rate increases further, the critical (or loaded) participation rate drops. This phenomenon is attributed to the following two nonlinear relationships: (i) for the sets of parameter values considered in these examples, as σ_r increases, the value of the call option in Equation (5.5) increases for $\rho \geq 0$ and decreases for $\rho < 0$; (ii) although the implied yield curves are upward sloping regardless of the volatility of the interest rates, the corresponding spot rates decline as the interest rates become more volatile.

These two properties lead to the empirical behavior of the critical participation rates. For $\rho < 0$, there are two contradicting effects simultaneously influencing the value of an EIA. Property (i) leads to a decline in the value of the option as interest rate volatility increases. At the same time, property (ii) implies that the spot rates will drop, inflating the expected value component involving the minimum guarantee term in equation (5.5). The net impact is that, with an initial decrease in the volatility of the interest rate, the first effect dominates and, hence, raises the participation rate to ensure that relation (4.2) (or 4.4) remains valid. However, as the interest rate continues to be more volatile, the second effect becomes more pronounced and, consequently, leads to a decrease in the fair participation rate.

In Table 2, we consider the impact on the critical and loaded participation rates by exerting a ceiling on the rates that can be credited. Comparing to a design without a cap, the presence of a cap should result in an increase in the critical and loaded participation rates. Furthermore, the magnitude of the increment should be more pronounced for a more volatile market environment. These intuitions are confirmed by the results shown in Table 2, which are based on the same set of examples as in Table 1 except for the enforcement of a 20% cap rate.

As an illustration, consider a low-volatility case with $\sigma_S = 20\%$, $\sigma_r = 4\%$ and a high-volatility case with $\sigma_S = 30\%$, $\sigma_r = 8\%$. In these cases, the critical participation rates for $\rho = 30\%$ and a 3% minimum guarantee on 90% of the premium are 88.97% and 75.12%, respectively. By confining the crediting rate to a 20% cap rate, the critical participation rates for the low- and high-volatility cases are 97.25% and 104.24%, representing an increment of approximately 9% and 39%, respectively. Hence, in both cases, the critical participation rates increase but the magnitude of the compensation is significantly greater for the more volatile situation.

Besides the increase in the critical (or loaded) participation rates, there are other notable nonlinear effects on the implied participation rates. These effects can be explained by comparing equation (5.3) to equation (5.5) (or equivalently comparing equation 5.4 to equation 5.6). For instance, for the design without a cap, we observed that, as the correlation increases, the critical (or loaded) participation rate is nonincreasing (see Remark 1D). This is attributed to the nondecreasing function of the underlying call option as we increase the correlation. The presence of a cap complicates this behavior considerably. Observe from Table 2 that, as we increase the correlation, the critical (or loaded) participation rates can

Table 3

Impact on Critical and Loaded Participation Rates by Varying Cap Rates in a Term-End Point-to-Point Design with σ_S = 20%, σ_r = 4%, g = 3% and β = 100%

Cap Rates (%) ζ	Critical Participation Rates Using Equation (4.2)			Loaded Participation Rates Using Equation (4.4) with n = 20			Loaded Participation Rates Using Equation (4.4) with n = 100		
	Correlation			Correlation			Correlation		
	−30%	0	30%	−30%	0	30%	−30%	0	30%
∞	86.62	85.04	83.55	85.57	83.98	82.49	86.17	84.58	83.09
50	86.63	85.05	83.56	85.58	84.00	82.51	86.18	84.59	83.11
45	86.64	85.05	83.57	85.60	84.02	82.53	86.19	84.60	83.12
40	86.65	85.07	83.60	85.63	84.05	82.58	86.21	84.63	83.16
35	86.69	85.14	83.70	85.70	84.15	82.71	86.27	84.71	83.27
30	86.89	85.42	84.07	85.95	84.48	83.13	86.48	85.01	83.66
25	87.70	86.48	85.39	86.83	85.60	84.51	87.32	86.10	85.01
20	91.03	90.48	90.06	90.25	89.68	89.23	90.70	90.13	89.71
18	94.46	94.47	94.61	93.72	93.69	93.78	94.15	94.14	94.25
16	101.02	102.01	103.12	100.30	101.22	102.27	100.71	101.67	102.75
14	115.09	118.19	121.52	114.36	117.36	120.56	114.78	117.84	121.11
12	155.34	166.09	178.05	154.48	164.99	176.66	154.97	165.62	177.45

remain unchanged, decrease, or even increase. The nonlinearity arises because, with an upper bound on the crediting rate, its value depends on the difference of two call options (see equation 5.3), and the difference of two nondecreasing functions is not necessarily nondecreasing.

Table 3 considers the effect on the implied participation rates as we change the cap rates. These examples are based on $\sigma_S = 20\%$, $\sigma_r = 4\%$, and $\rho = -30\%, 0, 30\%$. First note that the results corresponding to $\zeta = \infty$ represent the design without the cap. Second, as we decrease the cap rates, the critical participation rates increase. The participation rates increase gradually, initially, but accelerate as the cap rates reduce further. For instance, consider the positively correlated case, where the critical participation rate increases by 5% when the cap rate drops from 20% to 18%. In contrast, when the cap rate reduces further, from 14% to 12%, the increment is approximately 47%. Third, it was pointed out that, as we raise the correlation, the participation rates can behave differently, depending on the set of parameter values. A similar situation is observed as we modify the cap rates.

The design in term-end point can be criticized on the ground that the rate of return process depends solely on the index level at the end of the term. In the event of a sudden adverse movement in the equity market, this feature can be devastating to investors. To mitigate the risk of depending on the index level on one particular day, an averaging scheme has been proposed. This design is commonly referred as the *Asian-end* since it depends on the daily, weekly, or monthly index levels in the year when the claim is surrendered. For instance, for an Asian-end indexing method with monthly averaging, the variable S^* in Equation (5.2) is defined as:

$$S^*(t) = \frac{1}{12}\sum_{k=0}^{11} S\left(t - \frac{k}{12}\right). \tag{5.8}$$

Since the averaging design is less volatile than the design that depends only on a single observation, this implies that the value of an EIA with Asian-end is cheaper than the corresponding EIA with term-end point.

Another more exotic structure that has appeared in EIAs is the *term-high* or *high water mark* design. In this case, the variable S^* corresponds to the highest realized index level over the term of the contract. The sampling frequency of the index levels is typically daily, monthly, or annually. Naturally, the higher the frequency of sampling, the more expensive the EIAs. Also, the high water mark design is more expensive than the corresponding EIAs with term-end point design, which, in turn, is more costly than the EIAs with Asian-end.

It should be emphasized that, although the variables S^* in both the Asian-end and high water mark designs are defined differently from that in the term-end point design, expressions (5.3) and (5.4), for EIAs without a cap, and expressions (5.5) and (5.6), for EIAs with a cap, are still applicable. Hence, the values of EIAs reduce to computing Asian options and lookback options for the Asian-end and high water mark designs, respectively. It is well known that a simple closed-form expression does not exist for the Asian options.[5] For the lookback options with discrete sampling points, values can be expressed in terms of the multivariate normal probabilities under the Black-Scholes framework with a constant interest rate assumption. Efficient approaches for handling this problem are discussed in Lin (1999) and Boyle, Lai, and Tan (2001). For our model, with the additional complexity that the interest rates are stochastic, we resort to simulation to compute the value of the EIAs.

In our numerical illustration, we also consider the same sets of parameter values as in Table 1. For each set of parameter values, we simulate 100,000 trajectories, where each trajectory corresponds to the joint processes $\{S(t)\}$ and $\{r(t)\}$ being simulated daily (assuming 252 trading days per year). The critical and loaded participation rates are then estimated from these trajectories using the bisection approach until equations (4.2) and (4.4) are satisfied. The above procedure is replicated independently 10 times to provide an estimate of the standard errors for the estimated implied participation rates. Tables 4 and 5 provide comparison on the critical and loaded participation rates for EIAs without cap and with Asian-end and high water mark designs, respectively. The values in the brackets are the corresponding estimated standard errors. The sampling frequency for both designs is monthly so that the S^* in Asian-end is defined according to equation (5.8), whereas in high water mark we have

$$S^*(t) = \max_{1\le k\le 12t} S\left(\frac{k}{12}\right). \qquad (5.9)$$

Observe that the critical (or loaded) participation rates for EIAs with Asian-end design are consistently higher than the respective rates in term-end point design, which, in turn, are higher than those from high water mark designs. This is consistent with the argument made earlier that, among the three designs, the Asian-end is the least expensive and the high water mark is the most expensive. Note that the degree of penalty in the critical (or loaded) participation for the high water mark design is large relative to both Asian-end and term-end point designs, particularly for more volatile cases. Apart from this, the behavior of the implied participation rates in these designs is very similar to those with term-end point design.

5.2 Annual Reset/Ratchet EIAs

In this subsection, we focus on the most popular type of EIAs known as the *annual reset* or *annual ratchet* EIAs. According to Marrion (2000, 2001), these products comprise nearly 70% of EIAs sold in the marketplace.[6] Generically, the payoff in year t for one unit of EIA is given by

$$C(t) = \max\left\{\prod_{s=1}^{t} \max[\min[1 + \alpha R_s - \gamma, 1 + \zeta], 1], \beta(1+g)^t\right\}, \qquad (5.10)$$

where the random variable R_s again measures the appreciation of the referenced index fund in year s, the yield spread γ is the flat rate that is deducted from the gain in the index fund each year. The remaining parameters α, ζ, β, and g are defined as in equation (5.1) or (4.1).

Variations of annual reset EIAs exist, depending on the precise definition of the random variable R_s. The most common type of design is based on averaging, which, according to Marrion (2000, 2001), represents approximately 60% of annual reset EIAs. One possible averaging approach is based on the monthly index levels, with $R(s)$ defined as

$$R_s = \frac{1}{S(s-1)} \frac{\sum_{k=0}^{11} S\left(s - \frac{k}{12}\right)}{12} - 1. \qquad (5.11)$$

[5] A closed-form solution for the Asian option exists but it is very complicated. It involves inversion of a nontrivial Laplace transform and is practically not feasible. See Geman and Yor (1993).

[6] Here we consider all EIAs with an annual resetting feature, including annual yield spread EIAs, which will be discussed at the end of this subsection.

Table 4

Point-to-Point with Asian-End Design (on Last 12 Months' Index)

Critical Participation Rates Based on Equation (4.2)				
σ_S	σ_r	$\rho = -30\%$	$\rho = 0$	$\rho = 30\%$
3% Minimum Guarantee on 100% Premium				
20%	0%	94.27 (0.11)	94.27 (0.11)	94.27 (0.11)
	4	94.89 (0.11)	93.28 (0.11)	91.76 (0.10)
	8	93.49 (0.12)	90.25 (0.11)	87.39 (0.10)
30	0	80.16 (0.15)	80.16 (0.15)	80.16 (0.15)
	4	80.64 (0.15)	79.32 (0.15)	78.08 (0.14)
	8	79.39 (0.16)	76.75 (0.15)	74.43 (0.13)
3% Minimum Guarantee on 90% Premium				
20	0	99.73 (0.11)	99.73 (0.11)	99.73 (0.11)
	4	100.25 (0.12)	98.98 (0.11)	97.75 (0.11)
	8	99.26 (0.12)	96.64 (0.11)	94.25 (0.10)
30	0	87.35 (0.16)	87.35 (0.16)	87.35 (0.16)
	4	87.82 (0.17)	86.65 (0.16)	85.55 (0.15)
	8	86.89 (0.17)	84.52 (0.16)	82.43 (0.15)

		Loaded Participation Rates Using Equation (4.4)					
		with $n = 20$			with $n = 100$		
σ_S	σ_r	$\rho = -30\%$	$\rho = 0$	$\rho = 30\%$	$\rho = -30\%$	$\rho = 0$	$\rho = 30\%$
3% Minimum Guarantee on 100% Premium							
20%	0%	93.67 (0.10)	93.67 (0.10)	93.67 (0.10)	94.01 (0.10)	94.01 (0.10)	94.01 (0.10)
	4	94.31 (0.11)	92.69 (0.10)	91.16 (0.10)	94.64 (0.11)	93.02 (0.10)	91.50 (0.10)
	8	92.93 (0.11)	89.67 (0.10)	86.81 (0.09)	93.25 (0.12)	90.00 (0.11)	87.14 (0.09)
30	0	79.27 (0.14)	79.27 (0.14)	79.27 (0.14)	79.77 (0.14)	79.77 (0.14)	79.77 (0.14)
	4	79.75 (0.15)	78.45 (0.14)	77.23 (0.13)	80.25 (0.15)	78.94 (0.14)	77.71 (0.14)
	8	78.53 (0.15)	75.92 (0.14)	73.63 (0.13)	79.02 (0.16)	76.39 (0.14)	74.08 (0.13)
3% Minimum Guarantee on 90% Premium							
20	0	99.51 (0.11)	99.51 (0.11)	99.51 (0.11)	99.63 (0.11)	99.63 (0.11)	99.63 (0.11)
	4	100.03 (0.12)	98.75 (0.11)	97.51 (0.11)	100.15 (0.12)	98.87 (0.11)	97.64 (0.11)
	8	99.03 (0.12)	96.39 (0.11)	94.00 (0.10)	99.15 (0.12)	96.53 (0.11)	94.13 (0.10)
30	0	87.00 (0.15)	87.00 (0.15)	87.00 (0.15)	87.19 (0.16)	87.19 (0.16)	87.19 (0.16)
	4	87.48 (0.16)	86.31 (0.15)	85.21 (0.15)	87.67 (0.16)	86.50 (0.16)	85.40 (0.15)
	8	86.56 (0.17)	84.20 (0.15)	82.11 (0.14)	86.74 (0.17)	84.38 (0.16)	82.29 (0.14)

Another popular design is the term-end point. In this case, the random variable R_s is solely determined by the index levels at the beginning and the end of year s through the following structure:

$$R_s = \frac{S(s)}{S(s-1)} - 1. \tag{5.12}$$

Comparing annual reset EIAs to point-to-point EIAs, the former can be more appealing to the investors for two reasons. First, the interest is credited each year for the annual reset EIAs. The credited interest cannot be lost even if the index subsequently goes down. Second, the index level used to determine the appreciation of the index is reset annually. This "lock in" feature can be extremely valuable, particularly in a more volatile market. For instance, let us consider an annual reset term-end point design and also assume that the index drops by 20% in the first year. By the nature of the design, no interest is credited in the first year, but the index appreciation for the second year would be based on the lower level. This implies that, even if the index returns to its initial level in the following year, the crediting rate would become 25% for that year. For these reasons, annual reset EIAs would be more expensive than the corresponding term-end point EIA. Note

Table 5
Point-to-Point with High-Water-Mark Design (Monthly Monitoring)

		Critical Participation Rates Based on Equation (4.2)		
σ_S	σ_r	$\rho = -30\%$	$\rho = 0$	$\rho = 30\%$
3% Minimum Guarantee on 100% Premium				
20%	0%	68.71 (0.06)	68.71 (0.06)	68.71 (0.06)
	4	69.06 (0.06)	67.89 (0.06)	66.77 (0.05)
	8	67.71 (0.07)	65.38 (0.06)	63.30 (0.05)
30	0	51.65 (0.07)	51.65 (0.07)	51.65 (0.07)
	4	51.83 (0.07)	51.05 (0.07)	50.32 (0.06)
	8	50.79 (0.08)	49.26 (0.07)	47.88 (0.06)
3% Minimum Guarantee on 90% Premium				
20	0	70.60 (0.06)	70.60 (0.06)	70.60 (0.06)
	4	70.87 (0.06)	69.87 (0.06)	68.91 (0.06)
	8	69.63 (0.07)	67.66 (0.06)	65.86 (0.05)
30	0	53.83 (0.07)	53.83 (0.07)	53.83 (0.07)
	4	53.97 (0.07)	53.29 (0.07)	52.65 (0.07)
	8	53.00 (0.08)	51.67 (0.07)	50.47 (0.06)

		Loaded Participation Rates Using Equation (4.4)					
		with n = 20			with n = 100		
σ_S	σ_r	$\rho = -30\%$	$\rho = 0$	$\rho = 30\%$	$\rho = -30\%$	$\rho = 0$	$\rho = 30\%$
3% Minimum Guarantee on 100% Premium							
20%	0%	67.72 (0.05)	67.72 (0.05)	67.72 (0.05)	68.28 (0.06)	68.28 (0.06)	68.28 (0.06)
	4	68.07 (0.06)	66.91 (0.06)	65.82 (0.05)	68.63 (0.06)	67.46 (0.06)	66.36 (0.05)
	8	66.74 (0.07)	64.46 (0.06)	62.42 (0.05)	67.29 (0.07)	64.98 (0.06)	62.92 (0.05)
30	0	50.75 (0.06)	50.75 (0.06)	50.75 (0.06)	51.26 (0.06)	51.26 (0.06)	51.26 (0.06)
	4	50.92 (0.07)	50.17 (0.06)	49.46 (0.06)	51.43 (0.07)	50.67 (0.07)	49.95 (0.06)
	8	49.91 (0.07)	48.43 (0.07)	47.10 (0.06)	50.41 (0.08)	48.90 (0.07)	47.54 (0.06)
3% Minimum Guarantee on 90% Premium							
20	0	69.73 (0.06)	69.73 (0.06)	69.73 (0.06)	70.22 (0.06)	70.22 (0.06)	70.22 (0.06)
	4	69.99 (0.06)	69.01 (0.06)	68.08 (0.05)	70.49 (0.06)	69.50 (0.06)	68.55 (0.05)
	8	68.78 (0.07)	66.85 (0.06)	65.09 (0.05)	69.26 (0.07)	67.31 (0.06)	65.53 (0.05)
30	0	53.02 (0.06)	53.02 (0.06)	53.02 (0.06)	53.48 (0.07)	53.48 (0.07)	53.48 (0.07)
	4	53.15 (0.07)	52.50 (0.07)	51.89 (0.06)	53.61 (0.07)	52.95 (0.07)	52.32 (0.06)
	8	52.21 (0.07)	50.93 (0.07)	49.77 (0.06)	52.65 (0.08)	51.35 (0.07)	50.17 (0.06)

that the values of the annual reset EIAs increase with the parameters β, g, α, ζ and decrease with the yield spread γ. This provides ways for reducing the cost of these annuities.

Similar to the point-to-point EIAs, we assess the cost of the annual reset EIAs by computing the implied participation rates from both equation (4.2) and equation (4.4) for a given set of parameters β, g, ζ, γ. Due to the complexity of the designs, we use simulation to estimate the implied rates. We consider three common types of EIAs. The first type is an annual reset EIA with a term-end point design and with $\zeta = \infty$, $\gamma = 0$, that is, no cap and no yield spread. The second type is similar to the first except with a ceiling of 20%. For the third case, we consider an annual reset EIA without a cap and yield spread, but with a monthly averaging indexing method; that is, the index gain is defined according to equation (5.11).

The results are reported in Tables 6, 7, and 8. The assumptions for the underlying index and interest rate processes are identical to those in the last subsection. The implied participation rates are also computed based on 10 independent replications with 100,000 trajectories in each sample. The behavior of the implied participation rates when changing the volatilities

Table 6
Annual Reset with Term-End Point Design (without Cap)

σ_S	σ_r	$\rho = -30\%$	$\rho = 0$	$\rho = 30\%$
		Critical Participation Rates Based on Equation (4.2)		
		3% Minimum Guarantee on 100% Premium		
20%	0%	64.12 (0.04)	64.12 (0.04)	64.12 (0.04)
	4	64.17 (0.04)	63.53 (0.04)	62.91 (0.03)
	8	62.98 (0.05)	61.73 (0.04)	60.58 (0.03)
30	0	49.36 (0.03)	49.36 (0.03)	49.36 (0.03)
	4	49.23 (0.04)	48.94 (0.03)	48.66 (0.03)
	8	48.23 (0.04)	47.66 (0.04)	47.12 (0.03)
		3% Minimum Guarantee on 90% Premium		
20	0	64.54 (0.04)	64.54 (0.04)	64.54 (0.04)
	4	64.57 (0.04)	63.98 (0.04)	63.43 (0.04)
	8	63.43 (0.05)	62.30 (0.04)	61.26 (0.04)
30	0	49.88 (0.03)	49.88 (0.03)	49.88 (0.03)
	4	49.73 (0.04)	49.48 (0.04)	49.24 (0.03)
	8	48.77 (0.04)	48.28 (0.04)	47.82 (0.03)

		Loaded Participation Rates Using Equation (4.4)					
		with n = 20			with n = 100		
σ_S	σ_r	$\rho = -30\%$	$\rho = 0$	$\rho = 30\%$	$\rho = -30\%$	$\rho = 0$	$\rho = 30\%$
		3% Minimum Guarantee on 100% Premium					
20%	0%	63.82 (0.03)	63.82 (0.03)	63.82 (0.03)	63.99 (0.03)	63.99 (0.03)	63.99 (0.03)
	4	63.87 (0.04)	63.23 (0.04)	62.61 (0.03)	64.04 (0.04)	63.40 (0.04)	62.78 (0.03)
	8	62.71 (0.05)	61.44 (0.04)	60.28 (0.03)	62.86 (0.05)	61.60 (0.04)	60.45 (0.03)
30	0	49.07 (0.03)	49.07 (0.03)	49.07 (0.03)	49.23 (0.03)	49.23 (0.03)	49.23 (0.03)
	4	48.95 (0.04)	48.66 (0.03)	48.37 (0.03)	49.11 (0.04)	48.82 (0.03)	48.54 (0.03)
	8	47.97 (0.04)	47.39 (0.04)	46.84 (0.03)	48.12 (0.04)	47.54 (0.04)	47.00 (0.03)
		3% Minimum Guarantee on 90% Premium					
20	0	64.37 (0.04)	64.37 (0.04)	64.37 (0.04)	64.47 (0.04)	64.47 (0.04)	64.47 (0.04)
	4	64.41 (0.04)	63.82 (0.04)	63.26 (0.04)	64.50 (0.04)	63.91 (0.04)	63.36 (0.04)
	8	63.30 (0.05)	62.15 (0.04)	61.11 (0.03)	63.37 (0.05)	62.23 (0.04)	61.19 (0.03)
30	0	49.72 (0.03)	49.72 (0.03)	49.72 (0.03)	49.81 (0.03)	49.81 (0.03)	49.81 (0.03)
	4	49.59 (0.04)	49.33 (0.04)	49.09 (0.03)	49.67 (0.04)	49.42 (0.04)	49.18 (0.03)
	8	48.64 (0.04)	48.14 (0.04)	47.68 (0.03)	48.71 (0.04)	48.22 (0.04)	47.76 (0.03)

of the index, interest rate, the correlation and across different types of designs are very similar to those observed in point-to-point EIAs. We summarize the notable differences in the following remarks.

REMARK 2A

The ratchet feature for the EIAs with term-end point design can be quite expensive. For example, for the term-end point-to-point design (without cap) with minimum guarantee applied to 100% of the premium, the critical participation rates are around 85% and 72% for σ_s = 20% and 30%, respectively. In contrast, by changing the crediting strategy to a ratchet design, the fair participation rates for the respective cases need to be reduced substantially to around 63% and 49%.

REMARK 2B

Unlike the point-to-point EIAs, the critical participation rates for the minimum guarantee on 90% of the premium are only slightly larger than the corresponding cases guaranteeing 100% of the premium. This suggests that whether one has the minimum guarantee on 90% or on the full premium, the associated cost is insignificant.

REMARK 2C

The impact for enforcing a ceiling on the crediting rate is of greater magnitude for the EIAs with

Table 7

Annual Reset with Term-End Point Design (with 20% Cap)

σ_S	σ_r	Critical Participation Rates Based on Equation (4.2) $\rho = -30\%$	$\rho = 0$	$\rho = 30\%$
3% Minimum Guarantee on 100% Premium				
20%	0%	93.25 (0.10)	93.25 (0.10)	93.25 (0.10)
	4	91.20 (0.12)	92.00 (0.11)	92.84 (0.11)
	8	86.94 (0.13)	88.14 (0.13)	89.50 (0.12)
30	0	100.46 (0.18)	100.46 (0.18)	100.46 (0.18)
	4	95.89 (0.21)	97.96 (0.21)	100.13 (0.20)
	8	87.58 (0.22)	90.81 (0.22)	94.36 (0.21)
3% Minimum Guarantee on 90% Premium				
20	0	94.15 (0.10)	94.15 (0.10)	94.15 (0.10)
	4	92.02 (0.12)	92.98 (0.12)	93.99 (0.11)
	8	87.81 (0.14)	89.35 (0.13)	91.06 (0.12)
30	0	102.44 (0.20)	102.44 (0.20)	102.44 (0.20)
	4	97.65 (0.22)	99.98 (0.22)	102.44 (0.22)
	8	89.23 (0.23)	92.95 (0.23)	97.06 (0.22)

		Loaded Participation Rates Using Equation (4.4)					
		with n = 20			with n = 100		
σ_S	σ_r	$\rho = -30\%$	$\rho = 0$	$\rho = 30\%$	$\rho = -30\%$	$\rho = 0$	$\rho = 30\%$
3% Minimum Guarantee on 100% Premium							
20%	0%	92.13 (0.09)	92.13 (0.09)	92.13 (0.09)	92.77 (0.10)	92.77 (0.10)	92.77 (0.10)
	4	90.19 (0.11)	90.91 (0.11)	91.67 (0.10)	90.76 (0.11)	91.53 (0.11)	92.34 (0.11)
	8	86.05 (0.13)	87.13 (0.12)	88.34 (0.11)	86.56 (0.13)	87.71 (0.12)	89.00 (0.11)
30	0	98.28 (0.17)	98.28 (0.17)	98.28 (0.17)	99.51 (0.18)	99.51 (0.18)	99.51 (0.18)
	4	93.99 (0.20)	95.89 (0.20)	97.90 (0.19)	95.06 (0.20)	97.06 (0.20)	99.16 (0.19)
	8	86.08 (0.21)	89.06 (0.21)	92.33 (0.20)	86.93 (0.21)	90.05 (0.21)	93.48 (0.20)
3% Minimum Guarantee on 90% Premium							
20	0	93.45 (0.10)	93.45 (0.10)	93.45 (0.10)	93.85 (0.10)	93.85 (0.10)	93.85 (0.10)
	4	91.39 (0.12)	92.30 (0.12)	93.26 (0.11)	91.75 (0.12)	92.69 (0.12)	93.67 (0.11)
	8	87.29 (0.13)	88.75 (0.13)	90.38 (0.12)	87.59 (0.13)	89.09 (0.13)	90.77 (0.12)
30	0	101.05 (0.19)	101.05 (0.19)	101.05 (0.19)	101.84 (0.19)	101.84 (0.19)	101.84 (0.19)
	4	96.46 (0.21)	98.68 (0.21)	101.02 (0.21)	97.13 (0.22)	99.41 (0.22)	101.82 (0.21)
	8	88.35 (0.22)	91.90 (0.22)	95.82 (0.22)	88.85 (0.22)	92.50 (0.23)	96.52 (0.22)

a reset feature. In some cases, the increase in the implied participation rates are almost double. This indicates that putting a ceiling on the credited rate significantly reduces the value of EIA and, hence, the policyholder needs to be compensated through a substantial increase in the participation rates.

REMARK 2D

By switching the indexing method from term-end point to averaging, the critical (or loaded) participation rates should increase, and this is consistent with our results (compare Table 1 to Table 4 and Table 6 to Table 8). The increment, however, is more significant for the annual reset EIAs.

So far, we have evaluated the costs of EIAs by comparing the implied participation rates. We now consider other ways of assessing the cost of EIAs. In theory, as long as all parameters but one are given, the remaining parameter value can always be implied from equation (4.2) (or 4.4). In the examples we have considered, we chose to deduce the fair participation rates. Alternatively, we could let the cap rate be the "free" parameter and fix the participation rate as well as the remaining parameters in equation (4.2) (or 4.4). The implied (critical or loaded)

Table 8
Annual Reset with Monthly Averaging Design (without Cap)

		Critical Participation Rates Based on Equation (4.2)		
σ_S	σ_r	$\rho = -30\%$	$\rho = 0$	$\rho = 30\%$
3% Minimum Guarantee on 100% Premium				
20%	0%	111.79 (0.04)	111.79 (0.04)	111.79 (0.04)
	4	111.57 (0.05)	110.81 (0.04)	110.09 (0.04)
	8	109.33 (0.06)	107.85 (0.05)	106.47 (0.04)
30	0	84.73 (0.03)	84.73 (0.03)	84.73 (0.03)
	4	84.27 (0.04)	84.03 (0.04)	83.80 (0.03)
	8	82.37 (0.05)	81.89 (0.04)	81.43 (0.04)
3% Minimum Guarantee on 90% Premium				
20	0	112.53 (0.04)	112.53 (0.04)	112.53 (0.04)
	4	112.29 (0.05)	111.61 (0.05)	110.97 (0.04)
	8	110.13 (0.06)	108.82 (0.05)	107.60 (0.04)
30	0	85.57 (0.04)	85.57 (0.04)	85.57 (0.04)
	4	85.09 (0.04)	84.90 (0.04)	84.72 (0.04)
	8	83.24 (0.05)	82.87 (0.05)	82.53 (0.04)

		Loaded Participation Rates Using Equation (4.4)					
		with $n = 20$			with $n = 100$		
σ_S	σ_r	$\rho = -30\%$	$\rho = 0$	$\rho = 30\%$	$\rho = -30\%$	$\rho = 0$	$\rho = 30\%$
3% Minimum Guarantee on 100% Premium							
20%	0%	111.25 (0.04)	111.25 (0.04)	111.25 (0.04)	111.56 (0.04)	111.56 (0.04)	111.56 (0.04)
	4	111.06 (0.05)	110.29 (0.04)	109.55 (0.04)	111.35 (0.05)	110.59 (0.04)	109.86 (0.04)
	8	108.86 (0.06)	107.35 (0.05)	105.95 (0.04)	109.13 (0.06)	107.63 (0.05)	106.25 (0.04)
30	0	84.25 (0.03)	84.25 (0.03)	84.25 (0.03)	84.52 (0.03)	84.52 (0.03)	84.52 (0.03)
	4	83.81 (0.04)	83.55 (0.04)	83.31 (0.03)	84.07 (0.04)	83.83 (0.04)	83.59 (0.03)
	8	81.93 (0.05)	81.43 (0.04)	80.96 (0.04)	82.18 (0.05)	81.69 (0.04)	81.23 (0.03)
3% Minimum Guarantee on 90% Premium							
20	0	112.24 (0.04)	112.24 (0.04)	112.24 (0.04)	112.41 (0.04)	112.41 (0.04)	112.41 (0.04)
	4	112.01 (0.05)	111.33 (0.05)	110.68 (0.04)	112.17 (0.05)	111.49 (0.05)	110.84 (0.04)
	8	109.90 (0.06)	108.57 (0.05)	107.33 (0.04)	110.03 (0.06)	108.71 (0.05)	107.48 (0.04)
30	0	85.30 (0.04)	85.30 (0.04)	85.30 (0.04)	85.45 (0.04)	85.45 (0.04)	85.45 (0.04)
	4	84.84 (0.04)	84.64 (0.04)	84.46 (0.03)	84.98 (0.04)	84.79 (0.04)	84.61 (0.03)
	8	83.02 (0.05)	82.64 (0.05)	82.28 (0.04)	83.14 (0.05)	82.77 (0.04)	82.42 (0.04)

cap rate always exists and is also unique, since the value of the EIA increases monotonically with the cap.

Table 9 produces the critical cap rates for annual reset EIAs with a term-end point indexing method. For brevity, the loaded cap rates are not reported. These examples assume full participation and the absence of a yield spread; that is, $\alpha = 1$ and $\gamma = 0$. The critical cap rates are consistent with the critical participation rates reported in Table 6. An immediate conclusion that can be drawn from this table is that the critical cap rate is insensitive to volatilities and correlation.

In all the previous examples, we have assumed $\gamma = 0$ so that nothing is deducted from the gain. We now consider the case with a nonzero yield spread. In fact, EIAs with a nonzero yield spread are commonly known as *annual yield spread* EIAs. Furthermore, the participation rates in these cases are typically 100%. EIAs of this design represent a market share of around 33% and are the second most popular EIAs currently being offered. As in the critical participation rate and the critical cap rate, we can similarly assess the cost of the annual yield spread EIAs by comparing the implied yield spread that satisfies equation (4.2). Note that, in this case, as

Table 9

Annual Reset with Term-End Point Design: Critical Cap Rates Implied from Equation (4.2)

σ_S	σ_r	$\rho = -30\%$	$\rho = 0\%$	$\rho = 30\%$
3% Minimum Guarantee on 100% Premium				
20%	0%	19.17 (0.01)	19.17 (0.01)	19.17 (0.01)
	4	18.87 (0.02)	19.01 (0.02)	19.14 (0.01)
	8	19.01 (0.02)	18.53 (0.02)	18.78 (0.01)
30	0	20.03 (0.01)	20.03 (0.01)	20.03 (0.01)
	4	19.70 (0.02)	19.85 (0.02)	20.01 (0.01)
	8	19.01 (0.02)	19.31 (0.02)	19.60 (0.02)
3% Minimum Guarantee on 90% Premium				
20	0	19.26 (0.01)	19.28 (0.01)	19.28 (0.01)
	4	18.97 (0.02)	19.13 (0.02)	19.28 (0.01)
	8	19.15 (0.02)	18.68 (0.02)	18.96 (0.02)
30	0	20.17 (0.01)	20.17 (0.01)	20.17 (0.01)
	4	19.83 (0.02)	20.00 (0.02)	20.17 (0.01)
	8	19.15 (0.02)	19.47 (0.02)	19.79 (0.02)

Note:

we increase the yield spread, the value of the EIA reduces. Tables 10 and 11 are based on the same indexing strategies as Tables 6 and 8, respectively. In these cases, the critical yield spreads are computed by letting participation rate α = 100% and cap rate $\zeta = \infty$. Based on these results, we make the following observations:

REMARK 3A

The critical yield spread is very sensitive to the volatility of the index. By increasing σ_S from 20% to 30%, the critical yield spread needs to be compensated more than double.

REMARK 3B

Averaging the index not only dampens its volatility, but also significantly reduces the critical yield spread. For instance, with a 30% volatility of the index, the critical yield spread drops from about 20% to 3% by switching the design from term-end to averaging.

Table 10

Annual Reset with Term-End Point Design: Critical Yield Spread Implied from Equation (4.2)

σ_S	σ_r	$\rho = -30\%$	$\rho = 0\%$	$\rho = 30\%$
3% Minimum Guarantee on 100% Premium				
20%	0%	9.11 (0.01)	9.11 (0.01)	9.11 (0.01)
	4	8.94 (0.02)	9.35 (0.01)	9.76 (0.01)
	8	9.27 (0.02)	10.12 (0.02)	10.95 (0.02)
30	0	21.49 (0.03)	21.49 (0.03)	21.49 (0.03)
	4	21.27 (0.03)	21.81 (0.03)	22.36 (0.03)
	8	21.72 (0.04)	22.84 (0.03)	23.95 (0.03)
3% Minimum Guarantee on 90% Premium				
20	0	8.80 (0.01)	8.80 (0.01)	8.80 (0.01)
	4	8.65 (0.02)	9.02 (0.02)	9.39 (0.01)
	8	8.94 (0.02)	9.70 (0.02)	10.44 (0.02)
30	0	20.58 (0.03)	20.58 (0.03)	20.58 (0.03)
	4	20.37 (0.03)	20.89 (0.03)	21.34 (0.03)
	8	20.75 (0.03)	21.74 (0.03)	22.72 (0.03)

Table 11

Annual Reset with Monthly Averaging Design (100% Participation Rate and No Cap): Impact of Correlation, Volatilities of Index and Interest Rates on the Critical Yield Spread Implied from Equation (4.2)

σ_S	σ_r	$\rho = -30\%$	$\rho = 0\%$	$\rho = 30\%$
3% Minimum Guarantee on 100% Premium				
20%	0%	−1.439 (0.004)	−1.439 (0.004)	−1.439 (0.004)
	4	−1.396 (0.005)	−1.330 (0.005)	−1.264 (0.004)
	8	−1.130 (0.007)	−0.987 (0.006)	−0.843 (0.005)
30	0	3.019 (0.008)	3.019 (0.008)	3.019 (0.008)
	4	3.086 (0.009)	3.174 (0.009)	3.262 (0.008)
	8	3.471 (0.012)	3.661 (0.010)	3.847 (0.009)
3% Minimum Guarantee on 90% Premium				
20	0	−1.505 (0.005)	−1.505 (0.005)	−1.505 (0.005)
	4	−1.459 (0.006)	−1.403 (0.005)	−1.348 (0.005)
	8	−1.204 (0.007)	−1.085 (0.006)	−0.967 (0.005)
30	0	2.782 (0.008)	2.782 (0.008)	2.782 (0.008)
	4	2.853 (0.010)	2.923 (0.009)	2.993 (0.008)
	8	3.211 (0.012)	3.362 (0.010)	3.509 (0.009)

REMARK 3C

Even with a volatility of 20% in the equity market, the critical yield spread for EIAs with averaging indexing method needs to be negative in order to satisfy equation (4.2). This implies that an EIA providing a 3% minimum interest guarantee and 100% participation in the index can be offered at no additional cost with the averaging design.

6. CONCLUDING REMARKS

In this paper, we introduced an economic model that captures not only the behavior of the equity index, but also the interest rates. This is an improvement over the traditional model, which only permits the index to be stochastic while having the interest rates constant. A pricing formula for EIAs is obtained using the risk-minimization hedging strategy. The pricing formula assumes that the mortality risk can be diversified and, hence, it might not be suitable for companies that sell a small number of EIA policies. We, thus, extend the pricing formula using the percentile premium principle. The new pricing formula then allows the addition of a margin for the mortality risk. Reserving with both formulas are briefly discussed.

A detailed numerical analysis is then performed for valuing various existing EIAs in the North American market. The results from this analysis may be used as guidelines and benchmarks for insurance companies for the valuation of their EIAs. Furthermore, our methodology may be used to evaluate variable annuities (VAs) because of the similarity in payoff structure between EIAs and VAs. As a caution, it should be noted that our conclusion may only be relevant to the types of EIAs we considered in this paper, particularly for the ranges of parameter values assumed. There are still some remaining issues related to the valuation of EIAs, which will be considered in our subsequent work. One important issue would be the identification of hedging strategies for these EIAs, involving bonds with various maturities and index options corresponding to the embedded guarantees.

We wish to point out that the early surrender/withdrawal risk is not discussed in this paper although it is important when pricing EIAs. The main difficulty of modeling the surrender/withdrawal risk is the lack of reliable data in this area. A common industrial practice is to assume a fixed percentage each year for the early surrender/withdrawal. In this case, our algorithms can easily be modified. However, it does not provide any new insight for the model and, therefore, has been omitted.

ACKNOWLEDGMENTS

Both authors are grateful to the Committee for Knowledge Extension and Research of the Society of Actuaries and the Natural Science and Engineering Research Council of Canada (NSERC) for financial support. The authors also thank J. Marrion for providing his survey on EIAs for 1999–2001, and to Professors Phelim Boyle and Elias Shiu and an anonymous referee for their valuable comments.

REFERENCES

AIT-SAHALIA, YACINE 1996. "Nonparametric Pricing of Interest Rate Derivative Securities," *Econometrica* 64(3): 527–60.

BOWERS, NEWTON L., JR., HANS U. GERBER, JAMES C. HICKMAN, DONALD A. JONES, AND CECIL J. NESBITT. 1997. *Actuarial Mathematics,* 2nd ed. Schaumburg, Ill.: Society of Actuaries.

BOYLE, PHELIM P., Y. Z. LAI, AND KEN SENG TAN. 2001. "Using Lattice Rules to Value Low-Dimensional Derivative Contracts," *9th International AFIR Colloquium Proceedings*, Vol. 2, pp. 111–34.

CHAN, K. C., G. ANDREW KAROLYI, FRANCIS LONGSTAFF, AND ANTHONY B. SANDERS. 1992. "An Empirical Comparison of Alternative Models of the Short-Term Interest Rate," *Journal of Finance* 47(3): 1209–27.

DUAN, JIN-CHAUN, AND JEAN GUY SIMONATO. 1999. "Estimating Exponential-Affine Term Structure Models by Kalman Filter," *Review of Quantitative Finance and Accounting* 13(2): 111–35.

FÖLLMER, HANS, AND DIETER SONDERMANN. 1986. "Hedging of Non-Redundant Contingent Claims," in *Contributions to Mathematical Economics,* pp. 205–23, edited by W. Hildenbrand and A. Mas-Colell. Amsterdam: North-Holland.

GEMAN, HÉLYETTE, AND MARK YOR. 1993. "Bessel Processes, Asian Options and Perpetuities," *Mathematical Finance* 3(4): 349–75.

HARDY, MARY R. 2003. *Investment Guarantees: Modeling and Risk Management for Equity-Linked Life Insurance.* Hoboken, New Jersey: Wiley & Sons, Inc.

HARRISON, J. MICHAEL, AND STANLEY R. PLISKA. 1981. "Martingales and Stochastic Integrals in the Theory of Continuous Trading," *Stochastic Processes and their Applications* 11(3): 215–60.

HULL, JOHN C. 2000. *Options, Futures, and Other Derivatives,* 5th ed. New York: Prentice Hall.

HULL, JOHN, AND ALAN WHITE. 1990. "Pricing Interest Rate Derivative Securities," *Review of Financial Studies* 3: 573–92.

JAMES, JESSICA, AND NICK WEBBER. 2000. *Interest Rate Modelling.* New York: John Wiley & Sons.

KLUGMAN, STUART A., HARRY H. PANJER, AND GORDON E. WILLMOT. 1998. *Loss Models: From Data to Decisions.* New York: John Wiley & Sons.

LIN, X. SHELDON 1999. "Valuation of Options on the Maximum/Minimum of Multiple Assets, Discrete Lookback Options and Equity-Indexed Annuities," *Finance* 20: 95–114.

MARRION, JACK. 2000. "Advantage 2000 Equity Index Report," (November). Online at http://www.indexannuity.org.

———. 2001. "Advantage March, 2001 Equity Index Report," (March). Online at http://www.indexannuity.org.

———. 2002. "2002 Third Quarter Index Annuity Sales." Online at http://www.indexannuity.org.

MØLLER, THOMAS. 1998. "Risk-Minimizing Hedging Strategies for Unit-Linked Life Insurance Contracts," *ASTIN Bulletin* 28(1): 17–47.

———. 2001a. "Hedging Equity-Linked Life Insurance Contracts," *North American Actuarial Journal* 5(2): 79–95.

———. 2001b. "Risk-Minimizing Hedging Strategies for Insurance Payment Processes," *Finance and Stochastics* 5(4): 419–46.

ØKSENDAL, BERNT K. 1998. *Stochastic Differential Equations: An Introduction with Applications.* 5th ed. Berlin: Springer.

RABINOVITCH, RAMON. 1989. "Pricing Stock and Bond Options When the Default-Free Rate is Stochastic," *Journal of Financial and Quantitative Analysis* 24(4): 447–57.

SCHWEIZER, MARTIN. 1991. "Option Hedging for Semimartingales," *Stochastic Processes and Their Applications* 37(2): 339–63.

———. 1995. "On the Minimal Martingale Measure and the Föllmer-Schweizer Decomposition," *Stochastic Analysis and Applications* 13(5): 573–99.

TIONG, SERENA 2000. "Valuing Equity-Indexed Annuities," *North American Actuarial Journal* 4(4): 149–70.

VASICEK, O. 1977. "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics* 5(2): 177–88.

Discussions on this paper can be submitted until January 1, 2004. The authors reserve the right to reply to any discussion. Please see the Submission Guidelines for Authors on the inside back cover for instructions on the submission of discussions.

vector, although the corresponding density generator may depend on n), one can automatically get out of the problem of inconsistency. Furthermore, we are pleased to find the nice asymptotically effective estimators for expectations, covariance matrices, and TCEs. We wish to emphasize only that the problem becomes essentially more complicated in the case when the covariance matrix does not exist. Then, in this case, the matrix S (using the notation of Professor Bilodeau) becomes an inconsistent estimator of the matrix Σ (up to multiplication by any constant), although the vector $\bar{\mathbf{x}}_n$ remains a consistent but very ineffective estimator of the location vector $\boldsymbol{\mu}$. In the case where the expectations do not exist, $\bar{\mathbf{x}}_n$ is already an inconsistent estimator of the vector $\boldsymbol{\mu}$. In the univariate case, the problem of estimating the location parameter μ has been discussed in Landsman and Youn (2003). Here the sample median has been suggested as an initial value, μ_0, in the iterative process of estimating μ. When the covariance matrix contains elements that are infinite, it appears that the problem of finding an effective estimator of Σ is not well documented in literature. However, one can suggest using the sample quantiles as a basis for the estimation (see Landsman 1996).

We again thank Professor Bilodeau for his discussion of the statistical estimation of TCEs and for providing the nice numerical example at the end. We would like to add a reference to Csörgõ and Zitikis (1996). In this paper the effective nonparametric estimation of the mean residual life functional, something closely related to TCEs, was considered.

REFERENCES

CSÖRGÕ, MIKLÓS, AND RICǍRDAS ZITIKIS. 1996. "Mean Residual Life Processes," *Annals of Statistics* 24(4): 1717–39.

LANDSMAN, ZINOVIY M. 1996. "Sample Quantiles and Additive Statistics: Information, Sufficiency, Estimation," *Journal of Statistical Planning and Inference* 52: 93–108.

LANDSMAN, ZINOVIY M., AND HEEKYUNG YOUN. 2003. "Credibility Formula for the Generalized Student Family." *Proceedings of the 7th International Congress of Insurance: Mathematics & Economics.* Online at http://isfa.univ-lyon1.fr/IME2003/cadres.htm.

"Valuation of Equity-Indexed Annuities under Stochastic Interest Rates," X. Sheldon Lin and Ken Seng Tan, October 2003

MARK D. J. EVANS*

The paper presents an analysis of equity index annuities reflecting the impact of stochastic interest. This is an important topic. The paper presents extensive formulaic and numerical development. I would like to add some comments regarding the interpretation of some of the results.

1. REMARK 1C

Remark 1c contains the comment, "This is to be expected because the more volatile the fund is, the greater the appreciation of the fund." While this statement is not likely to lead the careful reader astray, a more precise statement is, "This is to be expected because the more volatile the fund is, the greater the expected appreciation of the fund given that it exceeds the strike."

2. REMARK 1F

Remark 1f discusses the interaction between interest rate volatility and the correlation between interest rates and index returns. For the negatively correlated cases, the participation rates increase initially with the volatility of the interest rates. As the volatility of the interest rate increases further, the participation rate drops. There is a simple reason for this, which is not captured in the paper. There are two forces at work. First, negative correlation dampens volatility, thereby reducing option costs and increasing participation rates. This tends to be a first order effect and thus behaves in approximately a linear fashion.

The second force at work is the convexity of options. Mathematically, this corresponds to the second derivative of option price with respect to interest rate. From Taylor's series, this effect is proportional to the square of the change in interest rates. This can be seen easily from Table 1 where the correlation is 0. The difference in

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participation rates between 8% interest rate volatility and 0% interest rate volatility is approximately four times as large as the difference in participation rates between 4% interest rate volatility and 0% interest rate volatility. For the 4% interest volatility, the negative correlation dominates due to its first order (linear) characteristics. As the interest volatility increases, the convexity becomes more important due to its second order (quadratic) characteristics.

Interest rate convexity in options increases with the time to maturity. For longer maturities, option convexity is large compared to a bond with similar duration. This is because interest rates impact not only the present value of the payoff, but also the probability and amount of the payoff. In the options market, this is reflected in the difference between implied and actual volatility, which tends to increase for longer maturities.

The authors point out that in Table 2, the existence of a cap can cause participation rates to either rise of fall as correlation increases. This is also observable in Table 3. The authors attribute this to the properties of the differences between two non-decreasing functions. The observed phenomenon can be explained as follows. Increasing correlation corresponds to a higher overall volatility. An option price can be expressed as the integral of the distance from the strike multiplied by the probability that the index finishes at that distance from the strike. With higher volatility, this probability distribution becomes wider and flatter. Based on the assumptions used in the paper, the expected equity growth rate exceeds the guarantee.

Because of this, higher volatility increases the probability of finishing out of the money since the left-hand tail will be larger. Without a cap, this is overwhelmed by the impact of the wider right-hand tail. But as the cap becomes smaller, more of the right-hand tail effect is removed, causing the effect noted by the authors.

3. Asian-end

The paper notes that an Asian-end is cheaper than a point-to-point. This is actually caused by the shorter average length of time to the ending observation of the underlying. This explains why Asianing reduces costs more for the Annual Reset with monthly averages than on a seven-year point-to-point with the last twelve months averaged. In the first case, we have an average term of 6.5 months versus one year; and, in the second case, we have an average term of 66.5 months versus seventy-two months. Note that the relationships in the participation rates are similar to the relationships of average maturity.

4. Remark 2b

Remark 2b notes that based on the assumptions in the paper, the impact of the 90% versus 100% guarantee is small on annual reset EIAs. The impact of the guarantee on participation rates is more material with a lower interest rate assumption.

Concluding Remarks

The authors claim that early surrenders do not provide any new insight for the model. In fact, the minimum guarantees in equity index annuities provide the customer with an additional option. In a severely down market, the value of the option and the guarantee at the end of the term may be less than the minimum guarantee at that point. Furthermore, the company is likely to find the sum of their zero-coupon bond and option (or similar investments depending on investment strategy) to be worth less than the minimum guarantee at such a point. This has a moderate impact on the price of the option in many product designs, especially on longer-term options. The impact on annual resets is not likely to be material. This option within an option is obviously path dependent. Approaches to valuing such path-dependent options include trees and Monte Carlo simulation. Assumptions about the efficiency of customer behavior may also be considered.

Authors' Reply

The authors are grateful to Mr. Mark Evans for his very insightful remarks on our paper. His detailed analysis and explanation greatly complement the paper. Here we would like to provide additional clarification on the Asian-end design, an issue raised by Mr. Evans.

He argued that an Asian-end design is cheaper mainly due to a shorter average maturity. To support his argument, he used the following two contracts as reported in our paper: an annual reset with monthly averaging and a point-to-point with Asian end with last twelve-month averaging. In our view, a direct comparison of these two designs does not necessarily lead to an easily interpretable result.

First, the crediting strategy is very different for these two designs; one ratchets every year while the other only at the end of the term. Second, one considers averaging monthly (and reset yearly) while the other only averaging in the final twelve months. Switching from term-end point-to-point to term-end annual reset increases the cost of the EIA and decreases the critical participation rate (see Remark 2A or compare Table 1 to Table 6).

However, as we switch from term-end point to monthly averaging indexing strategy, this dampens the cost of the guarantee so that the annual reset with monthly averaging indexing should be cheaper relative to annual reset with term-end point (compare Table 6 to Table 8). Hence if we were to directly compare the annual reset with monthly averaging and a point-to-point with Asian end with last twelve-month averaging, it appears that there are two conflicting effects affecting the cost of the contracts.

Consequently, it is difficult to draw a definitive conclusion by comparing these two designs. This is also the reason that in our paper we only make comparison between Table 1 and Table 4 and between Table 6 and Table 8 (see Remark 2D). In any case, we believe that an Asian-end design is cheaper mainly because of the increasing number of sampling points, which reduces the volatility level of the guarantee due to the Law of Large Number. Since in our model the volatility level of the index is proportional to the maturity, our numerical results appear to support his argument.

"Credit Standing and the Fair Value of Liabilities: A Critique," by Philip Heckman, January 2004

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Philip Heckman advances a number of arguments in opposition to reflecting credit risk in the fair valuation of liabilities. Since these arguments continue to surface and are a source of concern to a number of people, I feel it is important to address each major argument with the goal of showing how these arguments may in fact lead to the wrong conclusion, in spite of their apparent logic.

ARGUMENT 1: LIABILITIES SHOULD NOT BE VALUED IN THE SAME MANNER AS ASSETS

In his article, Dr. Heckman cites another article by Crooch and Upton (2001). In reference to an example from this article, Dr. Heckman notes on page 72 that one result of fair valuation is that "[t]wo companies [that] undertake *identical* obligations . . . [would] post different liabilities." According to Dr. Heckman, "company A [which has better credit risk] suffers a penalty relative to company B [which has worse credit risk] *because* of its superior credit."

Dr. Heckman maintains that "[a]lthough the same financial and economic principles apply to the valuation of liabilities as to the valuation of assets, these principles emphatically do not lead to identical valuations for the same obligation considered as an asset and as a liability" (p. 72). He proposes that, instead, a liability should be valued solely on the basis of the contractual terms "*under the assumption that the contract will be performed as written and in the full amount, that is, as if it were default-free*" (p. 81).

ARGUMENT 1: AN ALTERNATIVE VIEW

The argument above reflects a misunderstanding of the nature of the products involved. In particular, a commitment to pay a fixed amount from a company with less capacity to pay this obligation is *not* the same as a commitment to pay the same fixed amount in the same time frame from a company with more capacity to pay. And this is true even if both companies have the same *intent* to pay their obligations ultimately. Therefore, these two commitments are not *identical* obligations (they involve different levels of risk), and this should be reflected in the pricing of each of these obligations.

Note that this is essentially the same concept as pricing a burger from an A-rated restaurant differently from a burger from a B-rated restaurant.

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