THE MANAGEMENT OF DECUMULATION RISKS IN A DEFINED CONTRIBUTION PENSION PLAN

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ABSTRACT

The aim of the paper is to lay the theoretical foundations for the construction of a flexible tool that can be used by pensioners to find optimal investment and consumption choices in the distribution phase of a defined contribution pension plan. The investment/consumption plan is adopted until the time of compulsory annuitization, taking into account the possibility of earlier death. The effect of the bequest motive and the desire to buy a higher annuity than the one purchasable at retirement are included in the objective function. The mathematical tools provided by dynamic programming techniques are applied to find closed-form solutions: numerical examples are also presented. In the model, the tradeoff between the different desires of the individual regarding consumption and final annuity can be dealt with by choosing appropriate weights for these factors in the setting of the problem. Conclusions are twofold. First, we find that there is a natural time-varying target for the size of the fund, which acts as a sort of safety level for the needs of the pensioner. Second, the personal preferences of the pensioner can be translated into optimal choices, which in turn affect the distribution of the consumption path and of the final annuity.

1. INTRODUCTION

The main difference between defined benefit (DB) and defined contribution (DC) pension plans is the way in which the financial risk is treated. In DB plans, the financial risk is borne by the sponsors of the plan, who do not know in advance what contribution rate will be needed to finance the benefits promised. In practice, this risk is usually borne by the employer: employees pay a fixed part of the total contribution, and the employer pays the remaining, and obviously aleatory, part of the adjusted contribution rate.

In DC plans, the financial risk is borne by the member: contributions are fixed in advance, and the benefits provided by the plan depend on the investment performance experienced during the active membership and on the price of the annuity at retirement, in the case that the benefits are given in the form of an annuity. Therefore, the financial risk can be split into two parts: investment risk, during the accumulation phase, and annuity risk, focused at retirement. To limit the annuity risk—which is the risk that high annuity prices (driven by low bond yields) at retirement can lead to a lower than expected pension income—in many plans the member has the possibility of deferring the annuitization of the accumulated fund. This possibility consists of leaving the fund invested in financial assets as in the accumulation phase and allows for periodic withdrawals by the pensioner, until annuitization occurs (if ever).

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The current actuarial literature about the financial risk in DB and DC pension plans is quite rich. Papers dealing with the typical risks in DB plans—for example, solvency risk and contribution rate stability risk—are, among others, Boulier, Trussant, and Florens (1995), Boulier, Michel, and Wisnia (1996), Cairns (2000), Haberman and Sung (1994), and Owadally and Haberman (2004). The financial risk in DC plans in the accumulation phase is considered, among others, by Blake, Cairns, and Dowd (2001), Booth and Yakoubov (2000), Boulier, Huang, and Taillard (2001), Haberman and Vigna (2002), Khorasanee (1998), and Knox (1993). Arts and Vigna (2003) and Chiarolla, Longo, and Stabile (2004) analyze both the accumulation and the distribution phase of a DC pension plan. The financial risk in the distribution phase of DC pension plans has been dealt with in many recent papers, including Albrecht and Maurer (2002), Blake, Cairns, and Dowd (2003), Gerrard, Haberman, and Vigna (2004), Gerrard et al. (2004), Kapur and Orszag (1999), Khorasanee (1996), Milevsky (2001), and Milevsky and Young (2002).

The focus of this paper is on the management of risks in the decumulation phase of a DC pension plan, and the assumption made is that the retiree takes the option of deferring the annuitization, meanwhile consuming some income withdrawn from the fund and investing the remaining fund. Such a pensioner has three principal degrees of freedom:

1. He or she can decide what investment strategy to adopt in investing the fund at his or her disposal
2. He or she can decide how much of the fund to withdraw at any time between retirement and ultimate annuitization (if any)
3. He or she can decide when to annuitize (if ever).

The three choices described may be affected (in the timing, or amounts, or both) by restrictions imposed by law or by the plan's rules. For example, in the United Kingdom, where the option is called the ''income drawdown option,'' the amount withdrawn must be between 35% and 100% of the annuity that would have been purchasable immediately on retirement, and annuitization of the fund must take place not later than age 75. On the other hand, there are no limitations on the asset allocation of the fund. In this paper we will sometimes refer to the phase after retirement as the ''drawdown phase'' in the case that the pensioner takes the option of deferring the annuitization of the fund.

The first two choices represent a classical intertemporal decision-making problem, which can be dealt with using optimal control techniques in the typical Merton (1971) framework, whereas the third choice can be tackled by defining an optimal stopping time problem.

In this paper, we apply the mathematical techniques provided by the theory of dynamic programming with the aim of outlining a decision tool that, applied properly, could help members of DC plans (or DC plans' advisors) in making their decisions regarding the first two of the three choices outlined above. The third choice—when to annuitize, analyzed with different approaches, for example, by Blake, Cairns, and Dowd (2003), Stabile (2003), Milevsky, Moore, and Young (2004), and Milevsky and Young (2004)—is the subject of ongoing research.

We consider the more general case in which the member can decide about both investment allocation and consumption. The particular case where the income withdrawn is fixed over time is also analyzed in Appendix A. In both cases, we allow for mortality in the model; therefore the possibility of bequeathing wealth in the case of death before annuitization becomes relevant for the investment/consumption choices, which consequently are affected by the importance given to the bequest motive.

Our paper differs from most of the others in that the consumption path is considered as a control variable available to the pensioner in the postretirement phase, whereas in most of the mentioned works the amount withdrawn consists of the exact amount that a level annuity bought at retirement would provide. The effect of choosing different (optimal) consumption paths is also analyzed, in a realistic setting. Milevsky and Young (2004) find also the optimal consumption over time, solving a similar investment/consumption problem, and also find the optimal time of annuitization. However, they do not include the bequest motive in their model, use a different utility function (namely, the power utility function), and do not follow a target-based approach to the decision-making problem of the pensioner.

This paper is organized as follows. In Section 2 the investment/consumption problem is presented and solved. In Section 3 the notion of ''natural target'' is introduced, and the solution derived in Section 2 is analyzed with this particular choice for the target pursued by the pensioner. Some comments on the problem with constraints are also made. Numerical examples are shown in Section 4. Section 5 concludes.

2. THE INVESTMENT/CONSUMPTION CHOICE PROBLEM FACED BY THE RETIREE

The retiree member of a DC pension plan acquires control of a fund that may be either used to purchase an annuity or invested in the financial market. Throughout the paper, the financial market will be described by the typical Black and Scholes framework: there is a risky asset and a riskless asset. The riskless asset has a constant force of interest, denoted by r. The price of the risky asset is assumed to follow a geometric Brownian motion model, that is, it evolves according to the following stochastic differential equation:

$$dQ(t) = \lambda Q(t)\, dt + \sigma Q(t)\, dW(t), \tag{2.1}$$

where $W(t)$ is a standard Brownian motion.

The pensioner withdraws from the fund an instantaneous amount of income $b(t)$ and invests a proportion of the portfolio in the risky asset equal to $y(t)$ at any time t. The stochastic differential equation that describes the evolution of the fund $X(t)$ is

$$dX(t) = [X(t)(y(t)(\lambda - r) + r) - b(t)]dt + X(t)y(t)\sigma dW(t). \tag{2.2}$$

We assume that the reasons that push the retiree to choose the option of deferring annuitization are both the hope of being able to purchase in the future an annuity higher than the pension income provided by immediate annuitization at retirement and the ability to bequeath wealth in the case of death before annuitization.

It seems reasonable to assume that the individual has a certain target in mind, which is pursued during the drawdown phase. In particular, we shall assume that the pensioner has two different kinds of targets: a target for the size of the fund and a desired level of income to be consumed. Deviations from the targets will result in a loss. Therefore, the loss experienced by the pensioner consists of a number of parts:

- A disutility continuously experienced when the income drawn down from the fund is below the ideal level of income; a similar disutility is experienced also if the income consumed is above the level that is considered necessary to the pensioner, in that consuming excessively may result in a high chance of failure in achieving the final target at the time of annuitization, and, even worse, may lead to a higher probability of eventual ruin
- A disutility arising whenever the level of the fund is below or above the target level at that time; imposing a penalty for deviations above the target fund can be explained by noting that allowing the fund to exceed the target level implies that the pensioner has exposed him- or herself to unnecessary risk in the past
- A terminal disutility engendered at the time T of annuitization by any discrepancy between the level of the annuity actually purchased and the ideal level set by the investor
- A positive utility experienced in the event of death before annuitization, because of the investor's ability to fulfil the motive of bequeathing the assets in the fund to a nominated individual.

We denote by b_0 the target level of income periodically withdrawn from the fund during the drawdown phase, by b_1 the target level of income from the annuity purchased at age T, and by $F(t)$ the running target for the level of the fund at age t. Typically there is a relationship between the income levels b_1 and b_0, namely, that the desired level b_1 after annuitization is likely to be greater than or equal to the target level of consumption b_0 during the drawdown phase (considering the fact that there may be restrictions during the decumulation phase and also the fact that medical expenses tend to increase at older ages). We can assume, without loss of generality, that

$$b_1 = \eta b_0.$$

The continuously experienced disutility can be written in terms of a loss function depending on the time, t, and the level of the fund, x:

$$L(t, x) = e^{-\rho t}[u(F(t) - x)^2 + v(b_0 - b(t))^2], \tag{2.3}$$

where u and v are nonnegative constants, interpretable as weights given to the desire to monitor the growth of the fund and the daily consumption, respectively, and ρ is the usual subjective intertemporal discount factor. The case where the income withdrawn over time is fixed at b_0 and the running loss consists only of the first term of equation (2.3) is developed in Appendix A.

As observed in Gerrard, Haberman, and Vigna (2004), the use of a quadratic loss function is not new in the context of pension funds. Some examples are Boulier, Trussant, and Florens (1995), Boulier, Michel, and Wisnia (1996), and Cairns (2000). From a theoretical point of view, the quadratic loss function penalizes any deviations above the target (as well as deviations below the target), and this can be considered as a drawback to the model. However, the choice of trying to achieve a target and no more than this has the effect of a natural limitation on the overall level of risk for the portfolio: once the target is reached, there is no reason for further exposure to risk, and therefore any surplus becomes undesirable. The idea that people act by following subjective targets is accepted in the decision theory literature. For example, Kahneman and Tversky (1979) support the use of targets in the cost function, and, more recently, Bordley and Li Calzi (2000) investigate and support the target-based approach to decision making under uncertainty. Another example of the use of targets in an insurance context is provided by Browne (1995), who derives optimal investment policies by minimizing the probability that the wealth hits a certain bottom level (ruin) before hitting a certain upper level (target).

In addition, as will be shown later in Section 3, with a proper and not unreasonable choice of the target, the fund never exceeds the target, and the optimal running consumption never exceeds the targeted consumption. Hence, the choice of a quadratic loss function can be justified, and it does have the advantage of leading to closed-form solutions.

The chosen disutility function, quadratic in the fund and in the consumption, is not new also in the literature of stochastic optimal control problems. In fact, it is a particular case of (stochastic) linear quadratic optimal control problems—problems with applications mainly in the engineering context—where the cost functional is quadratic in both the state variable and the control (see, e.g., Yong and Zhou [1999] for a detailed description of linear quadratic optimal control problems, with examples of applications).

The terminal cost that comes into operation in the event of survival to age T takes the form

$$K(x) = we^{-\rho T}(b_1 - kx)^2, \tag{2.4}$$

and the utility of bequeathing assets of x on death at age t is

$$M(t, x) = e^{-\rho t}nx. \tag{2.5}$$

The positive constant k can be seen as the amount of annuity provided by the insurance company at age T for one unit of capital, w and n are nonnegative constants, interpretable as weights given to the achievement of the final annuity level of b_1 and to the importance given to the ability to leave a bequest. In particular, the constant n is associated with the size of the fund at the time of death.

The term $u(F(t) - x)^2$ in the loss function can be interpreted as having two main aims: on the one hand, it can help with reaching the final annuity target; on the other hand, it serves as an incentive for the maintenance of a certain minimum level of fund for a bequest, in the case of earlier death. Therefore, it can be argued that it is redundant as the same goals are pursued by the final loss in terms of the deviations from the desired annuity (function $K(x)$) and by the utility at death in terms of the bequest (function $M(t, x)$). However, we think that the constant monitoring of the fund size over time has an importance by itself, in that pensioners can check the performance of the fund against predetermined targets, given that insurance companies normally would supply regular reports on the value of the fund. Furthermore, the importance of interim targets can be reduced by choosing a low value of u. On the other hand, it will be shown (see Section 3.1) that the utility attached to the ability to bequeath turns out to be redundant: the term $M(t, x)$ can be incorporated in the interim targets.

We notice that the utility associated with the bequest is linear in the wealth, whereas the other kinds of loss are quadratic. One reason for this choice is that although it is natural for the pensioner to pursue targets for the level of consumption and the size of the fund, and the choice of a quadratic loss function is intended to penalize any deviation from the targets (see discussion above), there seems to be no natural target for the size of bequest to be left to heirs, and in case of death before annuitization, the higher the fund the better. The analysis has been carried out also with a quadratic term nx^2 instead of the linear one nx: in this case, solutions can be found in closed form too.

The total expected loss from age t onward is

$$H_{t,x}(y(\cdot), b(\cdot)) = \mathbb{E}\left[\int_t^{T\wedge T_D} L(s, X(s))\, ds + K(X(T))1_{T_D>T} - M(T_D, X(T_D))1_{T_D<T} \middle| X(t) = x\right], \tag{2.6}$$

where T_D is the random time of death.

The objective is to minimize over possible investment and consumption choices the expected discounted future loss from retirement until time $T \wedge T_D$, and find the optimal value function:

$$V(t, x) = \min_{y(\cdot),b(\cdot)} H_{t,x}(y(\cdot), b(\cdot)) \tag{2.7}$$

and the optimal couple $(y^*(\cdot), b^*(\cdot))$ that satisfies $V(t, x) = H_{t,x}(y^*(\cdot), b^*(\cdot))$.

2.1 Solution of the Problem

Minimizing equation (2.7) is a stochastic optimal control problem. In solving the problem, we follow the method used in Gerrard et al. (2004), so the reader should refer to this and references therein for a detailed explanation of the derivation of the HJB equation. For a more general derivation of a Bellman system of differential equations in a problem when benefits are triggered by any transition between different states in a finite Markov chain, applied to an insurance policy, see Steffensen (2004).

The optimal value function $V(t, x)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} 0 = \min_{y,b} \{ & e^{-\rho t}\, [u(F(t) - x)^2 + v(b_0 - b)^2] + V_x[-(b - rx) + (\lambda - r)xy] \\ & + V_t + \tfrac{1}{2}\sigma^2 x^2 y^2 V_{xx} - \delta(t)V - \delta(t)e^{-\rho t} nx\} \end{aligned} \tag{2.8}$$

with the boundary condition

$$V(T, x) = K(x) = we^{-\rho T}(b_1 - kx)^2, \tag{2.9}$$

where V_t, V_x, and V_{xx} represent $\partial V/\partial t$, $\partial V/\partial x$, and $\partial^2 V/\partial t^2$, respectively, and $\delta(t)$ is the force of mortality of the individual at age t.

Minimizing over y and b we find that the optimal proportion of wealth to invest in the risky asset and the optimal income to draw down from the fund are given by the following optimal control functions:

$$y^*(t, x) = -\frac{(\lambda - r)}{\sigma^2 x} \frac{V_x}{V_{xx}}, \tag{2.10}$$

$$b^*(t, x) = b_0 + \frac{1}{2v} e^{\rho t} V_x. \tag{2.11}$$

Substituting these values into the HJB equation we obtain

$$\frac{\beta^2 V_x^2}{2V_{xx}} + \frac{e^{\rho t}}{4v} V_x^2 + (b_0 - rx)V_x - V_t + \delta(t)V = e^{-\rho t}\ [u(F(t) - x)^2 - \delta(t)nx],$$

where $\beta = (\lambda - r)/\sigma$, the Sharpe ratio of the risky asset.

As in previous work, we seek a solution of the form

$$V(t, x) = e^{-\rho t}\ (A(t)x^2 + B(t)x + C(t)). \tag{2.12}$$

This formulation works as long as $A(t)$, $B(t)$, and $C(t)$ satisfy the following system of differential equations

$$\begin{cases} A'(t) = \dfrac{1}{v} A(t)^2 + (\rho - 2r + \beta^2 + \delta(t))A(t) - u, \\ B'(t) = \left(\rho + \beta^2 - r + \delta(t) + \dfrac{A(t)}{v}\right) B(t) + 2A(t)b_0 + 2uF(t) + n\delta(t), \\ C'(t) = \dfrac{\beta^2 B(t)^2}{4A(t)} + \dfrac{B(t)^2}{4v} + b_0 B(t) + (\rho + \delta(t))C(t) - uF(t)^2 \end{cases} \tag{2.13}$$

with boundary conditions given by

$$A(T) = wk^2, \qquad B(T) = -2wkb_1, \qquad C(T) = wb_1^2. \tag{2.14}$$

Note that one requirement for the proposed solution to be optimal is that $V_{xx} > 0$, or in other words that $A(t) > 0$ for all $t \le T$. The differential equation for $A(t)$ is of Riccati type and can be solved easily if one particular solution is known. The diffculty in finding an analytical solution comes from the presence of the time-dependent term $\delta(t)$ in the coefficient of $A(t)$. Therefore, we will consider different cases for the specification of $\delta(t)$.

The difficulty of the task (solving a Riccati differential equation) arises when finding the solution in the linear stochastic regulator problem (see also Øksendal 1998). This is because of the quadratic term in the control variable (here, the consumption).

2.2 On the Solution of the Differential Equation for *A*(*t*): Constant Mortality

In the special case where $\delta(t) = \delta$ for all t, we are able to find closed-form solutions. In this case we have to solve the following problem:

$$\begin{cases} A'(t) = \dfrac{1}{v} A(t)^2 + \phi A(t) - u \\ A(T) = wk^2, \end{cases} \tag{2.15}$$

with $\phi = \rho - 2r + \beta^2 + \delta$.

The solution is

$$A(t) = \frac{f_1(wk^2 - f_2)e^{R(T-t)} - f_2(wk^2 - f_1)}{(wk^2 - f_2)e^{R(T-t)} - (wk^2 - f_1)},$$

using the notation

$$R = \sqrt{\phi^2 + 4\frac{u}{v}}, \qquad f_1 = \frac{v}{2}(R - \phi), \qquad f_2 = -\frac{v}{2}(R + \phi).$$

We observe that

1. $f_1 \ge 0 > f_2$.
2. $\lim_{t \to -\infty} A(t) = f_1$.

3. $A'(t)$ is positive if $A(t) > f_1$, negative if $f_2 < A(t) < f_1$. If $A(t_0) > f_1$, then $A(t) > f_1$ for all t, and in particular $A(T) > f_1$. Conversely, if $A(t_0) < f_1$, then $A(t) < f_1$ for all t, and in particular $A(T) < f_1$.
4. We can restate the same remarks from a different viewpoint. If $A(T) > f_1$, then it must have been the case that $A(t_0) > f_1$ and A increases over the range (t_0, T). If, on the other hand, $f_2 < A(T) < f_1$, then it must be the case that $A(t_0) < f_1$ and A is a decreasing function over the range (t_0, T), so it must be greater everywhere than $A(T)$, which is equal to wk^2, and therefore strictly positive.

As a consequence, $A(t) > 0$ for all $t_0 \leq t \leq T$.

2.3 On the Solution of the Differential Equation for *A*(*t*): Age-Dependent Mortality

In the more realistic case of nonconstant mortality, the problem to solve is

$$\begin{cases} A'(t) = \dfrac{1}{v} A(t)^2 + \phi(t)A(t) - u \\ A(T) = wk^2, \end{cases} \tag{2.16}$$

where $\phi(t) = \rho - 2r + \beta^2 + \delta(t)$.

Equation (2.16) is a Riccati differential equation, meaning that it may not be possible to write down a solution in explicit form for an arbitrarily chosen force of mortality $\delta(t)$. Some common forms for $\delta(t)$ are $\delta(t) = \phi + \psi t$ (linear), $\delta(t) = \phi e^{\omega t}$ (Gompertz), and $\delta(t) = \zeta + \phi e^{\omega t}$ (Makeham). In all these cases the solution $A(t)$ exists but can be written as a nonlinear combination of Whittaker functions (which are combinations of Kummer functions); therefore it cannot be easily treated in the numerical applications.

However, it is possible to investigate the most important property of $A(t)$—whether it is positive—without solving explicitly. In fact, it is possible to prove the following fact:

Proposition 1

If the force of mortality is bounded over the range $t_0 < t < T$, *then* $A(t) > 0$ *for all* $t_0 < t < T$.

The proof is in Appendix B.

REMARK

The force of mortality may be assumed to be a nondecreasing function of time over the range $t_0 < t < T$ (this is true given that we are considering postretirement ages, for example, $t_0 = 60$ and $T = 75$; it would not be necessarily true for other ranges of age, like the range 15–35 for the male population), and therefore may be assumed to be bounded above and below.

Furthermore, it is also possible to approximate rather well the solution in the applications. In fact, we prove the following lemma (the proof, which is needed in the proof of the proposition, is also in Appendix B):

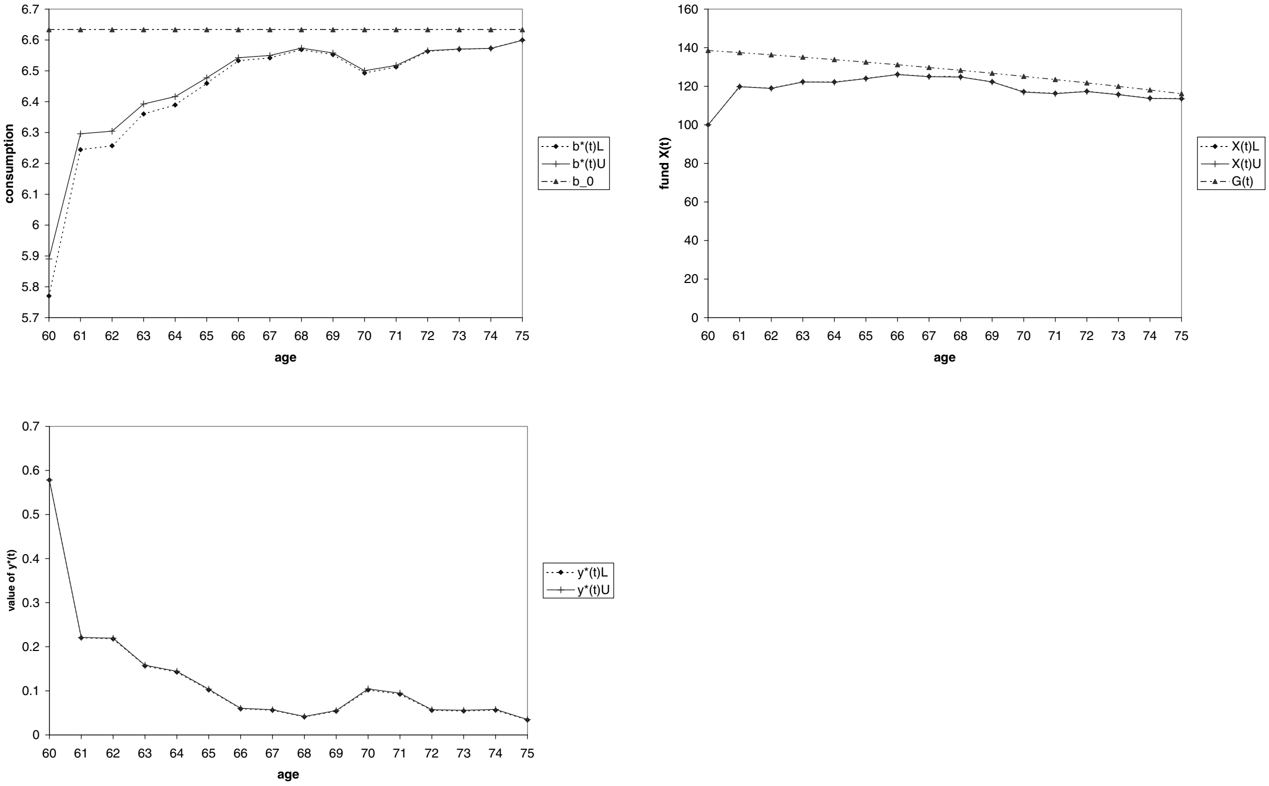
Table 1

Values of Functions $A_L(t)$ and $A_U(t)$

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$A_L(t)$	11.2	11	10.9	10.7	10.5	10.3	10	9.8	9.5	9.2	8.8	8.4	8	7.6	7.1	6.5
$A_U(t)$	9.6	9.5	9.5	9.3	9.2	9.1	8.9	8.8	8.6	8.4	8.1	7.9	7.6	7.3	6.9	6.5

Figure 1

Comparison between Optimal Controls and Growth of the Fund with Two Different Constant Forces of Mortality (at Age 60 and at Age 75)



Lemma 2

Suppose that $A_0(t)$ *solves equation (2.16) in the case where* $\phi(t) = \phi_0$*, a constant, and that* $A(t)$ *is a solution to equation (2.16) in the general case. If* $\phi(t) \geq \phi_0$ *for all* $t_0 \leq t \leq T$*, then* $A(t) \leq A_0(t)$ *for all* $t_0 \leq t \leq T$*. Conversely, if* $\phi(t) \leq \phi_0$ *for all* $t_0 \leq t \leq T$*, then* $A(t) \geq A_0(t)$ *for all* $t_0 \leq t \leq T$*.*

Therefore, if the real force of mortality $\delta(t)$ is bounded between a lower constant force of mortality δ_L and a higher constant force of mortality δ_U, the behavior of the true $A(t)$ will be bounded between the corresponding $A_L(t)$ and $A_U(t)$, solutions of the differential equation with δ_L and δ_U, respectively, and can be well approximated by these boundaries if these are close enough.

In the numerical applications of the model (presented in Section 5), the plot of the functions $A_L(t)$ and $A_U(t)$, corresponding respectively to $\delta_L = \delta_{t_0}$ and $\delta_U = \delta_T$ (with t_0 age at retirement, e.g., 60 and T age of compulsory annuitization, e.g., $T = 75$) shows that these functions are very close to each other, as are the corresponding optimal controls. An example is given by Table 1, which reports the values of these functions, and Figure 1, which illustrates the corresponding optimal consumption and investment choices and the resulting optimal growth of the fund, in one particular scenario of market returns. The forces of mortality at age 60 and at age 75 are calculated according to the Italian projected mortality table RG48 (males) data.

In the graphs reporting the optimal controls and the evolution of the fund under optimal control, the two curves corresponding to the different forces of mortality cannot be clearly distinguished, because they are almost coincident. This underlines the negligible effect of the precise value of the force of mortality in the practical applications considered here.

3. THE "NATURAL" TARGET FUNCTION

We now introduce a new function, which will turn out to be useful throughout the paper:

$$G(t) = -\frac{B(t)}{2A(t)}.$$

The optimal controls can be expressed as functions of $A(t)$ and $G(t)$:

$$b^*(t, x) = b_0 - \frac{A(t)}{v}(G(t) - x), \qquad y^*(t, x) = \frac{\lambda - r}{\sigma^2}\frac{G(t) - x}{x}. \tag{3.1}$$

Here we consider the properties of the optimally controlled process on the assumption that there is no restriction on the possible values taken by the controls b^* and y^*. The effect of imposing restrictions on the controls will be considered in a later section.

Let us denote by $X^*(t)$ the process of the fund under optimal control. Then

$$dX^*(t) = \left[-b_0 + \left(\frac{A(t)}{v} + \beta^2\right)(G(t) - X^*(t)) + rX^*(t)\right]dt + \beta(G(t) - X^*(t))\,dW(t).$$

As it appears that a pivotal quantity is the shortfall $G(t) - X^*(t)$, we shall denote this process by $S(t)$. We observe that

$$dS(t) = \left[G'(t) + b_0 - rG(t) - \left(\frac{A(t)}{v} + \beta^2 - r\right)S(t)\right]dt - \beta S(t)\,dW(t).$$

Although we have no explicit solution for $A(t)$, we nevertheless can write

$$G'(t) = -\frac{B'(t)}{2A(t)} + \frac{B(t)A'(t)}{2A(t)^2} = \left(r + \frac{u}{A(t)}\right)G(t) - b_0 - \frac{2uF(t) + n\delta(t)}{2A(t)}. \tag{3.2}$$

Substituting this back in, we obtain

$$dS(t) = \left(r - \beta^2 - \frac{A(t)}{v}\right)S(t)\,dt - \beta S(t)\,dW(t) + \frac{2u(G(t) - F(t)) - n\delta(t)}{2A(t)}\,dt. \tag{3.3}$$

The stochastic differential equation (3.3) satisfied by $S(t)$ suggests a natural form for the target function $F(t)$: if it is the case that

$$G(t) - F(t) = \frac{n}{2u}\delta(t), \tag{3.4}$$

then the logarithm of $S(t)$ is a Wiener process with a time-dependent drift, and therefore $S(t)$ will always remain nonnegative, under the assumption that $S(t_0) > 0$. In other words, if $X(t_0) < G(t_0)$, then $X(t)$ will always remain below the function $G(t)$.

This property was also observed for the model discussed in Gerrard, Haberman, and Vigna (2004), in a problem where consumption was fixed. In that case, it was discovered that there was a natural explanation for the target $F(t)$ that arose out of the equations, namely, that it consisted of precisely the amount of money required to fund consumption at the fixed level until the time of compulsory annuitization and then to achieve the final target pursued. Here we find a similar explanation, which includes also the bequest motive. In fact, it is easy to prove that the functions $F(t)$ and $G(t)$ which satisfy equations (3.2) and (3.4) are

$$G(t) = \frac{b_0}{r}(1 - e^{-r(T-t)}) + \frac{b_1}{k}e^{-r(T-t)}, \qquad F(t) = \frac{b_0}{r}(1 - e^{-r(T-t)}) + \frac{b_1}{k}e^{-r(T-t)} - \frac{n}{2u}\delta(t). \tag{3.5}$$

Even when F and G do not satisfy equation (3.4), it is still possible to obtain a solution to equation (3.2), at least in the form

$$G(t) = \frac{b_1}{k} \exp\left(-r(T-t) - u\int_t^T \frac{ds}{A(s)}\right)$$
$$+ \int_t^T \left(b_0 + \frac{2uF(z) + n\delta(z)}{2A(z)}\right) \exp\left(-r(z-t) - u\int_t^z \frac{ds}{A(s)}\right) dz. \tag{3.6}$$

The interpretation for the choice of the natural target $F(t)$ is the following. If a sum $G(t)$ were invested at time t in the risk-free asset, then the interest payments would cover consumption at rate b_0 until the age of compulsory annuitization, and thereafter would permit the purchase of an annuity paying the required amount b_1 per unit time. Therefore, the level $G(t)$ can be considered to be a sort of "safety level" for the personal needs of the pensioner (see Gerrard, Haberman, and Vigna 2004) and in effect would coincide with his or her target, should he or she have no bequest motive (i.e., $n = 0$). If the pensioner has a bequest motive, his or her target would be chosen according to the importance given to it, and we may think of the quantity $n/2u\ \delta(t)$ as the part of the overall target pursued by the pensioner that allows for the bequest desire. In fact, it will be shown in the next section that, in the case of the natural target, the problem to be solved is equivalent to one in which the interim target is equal to the safety level $G(t)$ and there is no bequest motive.

3.1 Interaction between the Bequest Motive and the Interim Target

This section arises from the observation that, surprisingly, when the natural target is chosen, the optimal controls do not depend on the weight given to the bequest motive, as we can see by observing that neither $A(t)$ nor $G(t)$ depend on the value of n (see eqs. (2.13), (3.1), and (3.5)), while they do depend on the force of mortality. It seems that if the pensioner does take into account the bequest motive and the subjective probability of death when selecting the target pursued (by appropriate choice of $F(t)$), his or her optimal behavior is then the same as if the bequest motive were absent.

This is indeed a special case of a more general fact that can be shown concerning the interaction between the choice of the interim target and the bequest motive. It can be shown that the utility attached to the bequest motive $M(t, x)$ (eq. (2.5)), from a purely mathematical point of view, is not actually needed in the formulation of the problem, in that the importance given to the bequest motive can be included in the interim targets. In fact, consider an individual with a certain target function $F(t)$ and a certain bequest motive given by $M(t, x)$. We can replace the interim target F with a new one, which includes the bequest motive:

$$\tilde{F}(t) = F(t) + \frac{n\delta(t)}{2u}. \tag{3.7}$$

If we replace equation (3.7) in equation (2.8), the Hamilton-Jacobi-Bellman equation becomes

$$0 = \min_{y,b}\left\{e^{-\rho t}\left[u(\tilde{F}(t) - x)^2 + v(b_0 - b)^2 + \frac{n^2\delta^2(t)}{4u} - n\delta(t)\tilde{F}(t)\right]\right.$$
$$\left. + V_x[-(b - rx) + (\lambda - r)xy] + V_t + \tfrac{1}{2}\sigma^2 x^2 y^2 V_{xx} - \delta(t)V\right\}. \tag{3.8}$$

In other words, the term $M(t, x) = -nx\delta(t)e^{-\rho t}$ has been replaced by the term $e^{-\rho t}[n^2\delta^2(t)/4u - n\delta(t)\tilde{F}(t)]$, which does not contain x, y, or b and therefore does not affect the optimal controls. It is clear that the two formulations of the problem (3.8) and (2.8) are equivalent, in that they give the same optimal control rules: in the new formulation the weight given to the bequest motive n still enters the optimal controls through the target itself $\tilde{F}$. We can think of the $\tilde{F}$ target as a target adjusted to include the bequest, and we can refer to it as a "bequest target," to distinguish it from the original one and the natural one. We notice that the bequest target increases as n increases and when $\delta(t)$ increases. This is reasonable: an individual who attaches high importance to the ability to leave a bequest and has a high subjective probability of imminent death will be likely to set a high interim target.

When the natural target is chosen, the bequest target is

$$\tilde{F}(t) = F(t) + \frac{n\delta(t)}{2u} = G(t).$$

Thus, in this case the bequest target coincides with the safety level. More important, we note that, since $\tilde{F}$ is the only way by which the weight n can affect the optimal controls, in this case the optimal controls are not affected by n, observing that the safety level G does not include n.

We notice that the absence of an effect of the bequest motive on optimal choices occurs only in the presence of the natural target: we can think of it as having been offset by the interim target. Should the pensioner choose a target other than the natural one, the optimal control would depend on n via $G(t)$ (see eq. (3.6)) or, in the new formulation, via $\tilde{F}$ (see eqs. (3.7)–(3.8)). We shall ignore this case in the rest of the paper and defer the analysis of other choices of the target to future research.

Although we have just proved that the explicit presence of a bequest motive in the form of equation (2.5) is not necessary, because the importance given to the bequest can be encompassed in the interim target, we nevertheless leave the formulation of the initial problem unchanged, so as to isolate the effect of the different components on the pensioner's behavior.

3.2 Some Features of the Optimal Controls When Natural Targets Are Chosen

The optimal control policies at time t, with a fund of $X^*(t)$, are the following:

$$b^*(t, X^*(t)) = b_0 - \frac{A(t)}{v}(G(t) - X^*(t)), \qquad y^*(t, X^*(t)) = \frac{\lambda - r}{\sigma^2}\frac{G(t) - X^*(t)}{X^*(t)}.$$

When the natural targets are chosen, the shortfall of the fund from $G(t)$ is always strictly positive. Therefore, the optimal consumption never exceeds the level b_0, and the amount invested in the risky asset is always positive. This property can be of considerable importance, given the fact that consumption can be limited by regulation (in the United Kingdom the amount withdrawn must lie between 35% and 100% of the amount of annuity provided by the fund at retirement), and short selling is likely to be forbidden, and given also the fact that adding constraints to the optimization problem would considerably increase the diffculty of finding and treating the solution (see below). Examples of works where an optimization problem with constraints has been solved are Di Giacinto and Gozzi (2004), in the context of a DC pension plan, and Browne (1995), who minimizes the probability of ruin when borrowing is not allowed.

We notice that the optimal amount invested in the risky asset $y^*(t)X^*(t)$ is proportional to the shortfall $S(t)$, which is the difference between the safety level and the fund level. This result is similar to a result found by Browne (1997): solving two "survival problems" (maximizing the probability of reaching a "safe region" before occurrence of ruin and minimizing the discounted penalty paid upon going bankrupt) he finds that in both problems the optimal policy implies investing in the risky asset a proportion of the (positive) difference between the amount needed for being in the safe region and the fund level.

The function $A(t)$ depends on the value of the parameters v, u, and w, but it can be seen from the form of the controls that what counts most is the ratio $A(t)/v$. From a detailed study of the solution $A(t)$ of the Riccati differential equation in the case of constant mortality, we see that

$$\lim_{v \to +\infty} \frac{A(t)}{v} = 0, \tag{3.9}$$

a result that can be extended naturally to the general case. This feature is also intuitive: when the importance attached to the monitoring of the consumption is high, the optimal consumption tends to coincide with the desired level b_0.

From numerical examples that will be shown later, we also see that this ratio depends heavily on the ratio u/v, rather than on the individual values of the parameters. Mathematically, this could be explained by observing that for points in time sufficiently far from time T (i.e., for $T - t$ sufficiently large), we have

$$\frac{A(t)}{v} \sim \frac{1}{2}(R - \phi),$$

recalling that $R = \sqrt{\phi^2 + 4(u/v)}$ and that ϕ does not depend on the parameters v, u, and w. When time T approaches, the ratio w/v becomes more important (since $A(T)/v = w/v\, k^2$). In the numerical applications, we have observed that the most significant factor driving the controls is the ratio u/v, which gives an indication of the relative importance attached to the monitoring of the fund and of the running consumption.

3.3 Imposing Restrictions on the Controls

Governments (or regulators) may introduce regulations to restrict the freedom of investors with regard to either the income they draw down or the proportion of the fund that may be invested in risky assets. To increase the generality of the treatment, however, we will consider only the natural restrictions that follow from the situation being modeled:

1. $b \geq 0$. Although it is not impossible to imagine negative consumption—rather than withdrawing money from the fund for day-to-day expenses, the investor pays additional sums into the fund—the protected status of pension funds is likely to rule this out as a possibility.
2. $y \leq 1$. Again, in normal circumstances an investor may well have the option of borrowing at a fixed rate of interest in order to invest in the risky asset, but in the particular context of a pension fund it is unlikely that this will be permitted.
3. $x \geq 0$. If the assets of the fund drop to zero, the rules of the fund presumably will require that the investor stops trading.

If the evolution of the fund is governed by the optimal controls derived with natural targets, then

$$b^*(t, X^*(t)) < 0 \Leftrightarrow S(t) > \frac{vb_0}{A(t)},$$

$$y^*(t, X^*(t)) > 1 \Leftrightarrow S(t) > \frac{G(t)}{1 + \dfrac{\lambda - r}{\sigma^2}},$$

$$X^*(t) < 0 \Leftrightarrow S(t) > G(t).$$

It is clear that $X^*(t) < 0$ can occur only if $y^*(t, X^*(t)) > 1$, so the second restriction will take effect more frequently than the third. No such relationship can be proved for the first two restrictions, or for the first and the third restrictions, however: which one takes effect first depends on the relative values of the weighting parameters u, v, and w. Namely, the critical level at which consumption becomes negative does depend on the choice of the parameters v, u, and w, whereas the critical levels for ruin and borrowing money from the bank do not. This means that, by choosing appropriate values of the parameters, negative consumption in practice can be avoided (it would be sufficient, for example, to choose values of the parameters for which ruin occurs before negative consumption). In fact, increasing the value of v will result in a higher level for the barrier for negative consumption (recalling the limit (3.9)). Unfortunately, there are no values of the parameters that would help in avoiding ruin or borrowing money from the bank.

The graphs in Appendix C (Figure A) show how the change in the ratio v/u can affect the level of the barrier for negative consumption. In particular, it can be seen that with a high enough v/u (e.g.,

160) the barrier for negative consumption is higher than the barrier for ruin: that is, ruin occurs before negative consumption.

3.3.1 The Case $b^* < 0$

If $S(t) > vb_0/A(t)$, the optimal choice of b is negative. If restriction 1 is in force, it would be natural to suggest that we choose zero consumption in such cases. It should be noted that such a strategy does not lead to the optimal control of the process subject to restriction 1, but the difference may be small.

If we are constrained to choose $b = 0$, the control problem becomes one of choosing y. This is related to the situation considered in Appendix A, where the income rate is required to take a fixed value b_0. The remarks made in Appendix A apply to this case, by taking $b_0 = 0$.

3.3.2 The Case $y^* > 1$

If $S(t) > G(t)/(1 + (\lambda - r)/\sigma^2)$, the optimal choice of y is greater than 1. If restriction 2 is in force, so that no bank borrowing is permitted, then the simplest suggestion is that we should take $y^* = 1$ in these cases. If we adopt this strategy, then the form of the HJB equation is substantially altered, being now

$$0 = -\frac{1}{4v} e^{\rho t} V_x^2 + u e^{-\rho t}(F(t) - x)^2 + (\lambda x - b_0)V_x + V_t + \tfrac{1}{2}\sigma^2 x^2 V_{xx} - \delta(t)v - \delta(t) n x e^{-\rho t}.$$

It is possible to find a solution to this equation, again of the form $V(t, x) = e^{-\rho t}(A(t)x^2 + B(t)x + C(t))$, but it is different from the solution that applies in the region $y^* < 1$.

3.3.3 The Case Where Both $b^* < 0$ and $y^* > 1$

When $X(t)$ is sufficiently low, both b^* and y^* fall outside the range permitted by the restrictions. Again a first approximation to the optimal strategy is to set $b = 0$, $y = 1$ in such cases. This implies that the entire fund is invested in the risky asset but that no income is drawn down. The fund, therefore, is a constant multiple of the price of the risky asset. As a geometric Brownian motion with positive drift, it is unable to become negative.

As a consequence, ruin is impossible when the restrictions are applied as long as $X(t)$ is such that $b^* < 0$ and $y^* > 1$.

4. NUMERICAL EXAMPLES

The model outlined in the previous sections has been tested in simulated scenarios for market returns. In particular, the path of the risky asset has been simulated for 1,000 scenarios via Monte Carlo simulations, and for each scenario the optimal policies (i.e., the optimal consumption and the optimal asset allocation) have been applied. The motivation for testing the model in simulated scenarios is to obtain extra information about many relevant issues when the optimal choices derived above are applied. These issues include the undesirable events listed above (ruin, negative consumption, and borrowing money from the bank), together with some information about the final outcome of the option of deferring annuitization, in terms of the size of the final annuity that can be bought at time T. So we are interested in investigating the following key features:

1. Risk of outliving the assets before time T, called the ruin probability, and average time of ruin, when ruin occurs
2. Behavior of the optimal consumption and the optimal investment allocation over time
3. Probability of negative consumption, average age at the time of negative consumption, and average time spent consuming negative amounts
4. Probability of borrowing money from the bank for investing in the risky asset, average age at the time of borrowing money, and average time spent borrowing money

5. Distribution of the final annuity that can be bought at time T
6. Probability that there is some time before T when the pensioner is able to buy a better annuity than the one that he or she could have bought at retirement, and comparison with the targeted annuity
7. Effect on the optimal controls and on the final annuity of the choice of the relative importance given to the running consumption and to the achievement of the target.

4.1 Assumptions and Methodology

In the simulations, we have chosen the natural target specification. The assumptions made on the parameters are the following:

- Retirement is at age $t_0 = 60$ and age of compulsory annuitization is $T = 75$
- The fund at retirement is $X(t_0) = 100$
- The amount of consumption targeted during the drawdown phase, b_0, is the level annuity that can be bought at age 60 with a fund of 100, adopting the Italian projected mortality table RG48 males, assuming the same interest rate as used for the riskless asset and a loading factor of 5%; thus, $b_0 = 6.63$
- The target for the final annuity is $b_1 = 1.5b_0 = 9.95$ and $b_1 = 2b_0 = 13.26$, to test different risk attitudes (the higher the target, the lower the risk aversion)
- Parameters for the asset returns are $r = 4\%$, $\lambda = 10\%$, $\sigma = 20\%$; the Sharpe ratio of the risky asset is therefore 0.3; the intertemporal discount factor is given by $\rho = 4\%$
- The boundaries chosen for the force of mortality are $\delta_L = \delta(60) = 0.004625$ and $\delta_U = \delta(75) = 0.026254$ (derived from the survival probabilities at ages 60 and 75 of the RG48 males, assuming a constant force of mortality over the year of age)
- The price of the final annuity at age 75 is $k^{-1} = a_{75}(1 + L)$, where L is the loading factor adopted by the company, chosen to be equal to 5%; the annuity at age 75 has been calculated with the table RG48 males
- The ratio v/u has been chosen to be equal to 10, 50, 100, and 500, the ratio w/v equal to 1 and 100, with a fixed $u = 1$; the weight given to the bequest motive (although it does not influence the optimal controls) has been chosen to be equal to $n = 10$.

Regarding the choice of the value of the parameters, we refer the reader to Section 4.5, where some indications are given.

The discretization of the process over the 15 years of the decumulation phase has been done on a weekly basis; for each combination of b_1, $\delta_{\tau=L,U}$, v/u, and w/v, 1,000 simulations have been run, using the same 1,000 streams of pseudo-random numbers for each combination (to allow consistent comparisons between different combination of parameters). In each scenario we have simulated the Brownian motion and hence the behavior of the optimal controls, as well as the evolution of the fund under optimal control. The distribution over the 1,000 simulations of $y^*(t)$ and $b^*(t)$, for $t = 0, 1, \ldots, 779$ (780 being the number of weeks in 15 years) has been analyzed through some relevant statistics, such as minimum, maximum, mean, standard deviation, and some percentiles (5th, 25th, 50th, 75th, and 95th). The distribution over the 1,000 simulations of the final annuity purchasable at time T with the resulting fund has been examined using the same statistical analysis, and the characteristics listed above have been investigated by checking the path of $y^*(t)$, $b^*(t)$, and $X^*(t)$ over time.

4.2 Simulation Results: Optimal Controls

We have found that the choice of the ratio w/v does not significantly affect results, therefore we show only results for the case of $w/v = 1$. Similarly, we present results only for the higher force of mortality $\delta_U = \delta(75)$, the corresponding results for the lower force of mortality $\delta_L = \delta(60)$ being almost identical (see, e.g., Figure 1). On the other hand, the dependence on the ratio v/u is quite strong, both for the optimal controls and for the distribution of the final annuity.

The graphs in Figure 2 show the median of the optimal investment in the risky asset $y^*(t)$ and the mean of the optimal consumption $b^*(t)$ over time, with the four choices of v/u and with the two different targets b_1.

Increasing the weight given to the running consumption (i.e., the value of v) results in increasing the optimal consumption $b^*(t)$: when $v/u = 500$, the individual consumes very close to the ideal level b_0 (plotted in the graph to allow comparisons). This results also in riskier investment strategies, as the level of the (median of) $y^*(t)$ is higher when v/u increases. However, with low values of v/u, the optimal policy would imply consuming small amounts of money at the beginning of the plan, or even consuming negatively (it turns out that this would happen only for a relatively short period of time after retirement; see later tables).

When increasing the final target from $b_1 = 1.5b_0$ to $b_1 = 2b_0$, the optimal investment allocation becomes riskier, and the optimal consumption decreases. This behavior of the optimal controls is intuitive, in that the final target has increased while everything else has remained unchanged.

The choice of the weights is relevant to the optimal consumption. The graphs in Figure 3 show min, max, and some percentiles of the optimal consumption over time with $v/u = 10$ and 100, with the two different final targets.

With $v/u = 10$, the initial level of optimal consumption is very close to 0 for $b_1 = 1.5b_0$ and negative for $b_1 = 2b_0$; furthermore the optimal consumption in 5% of the cases is negative for at least one and a half years when $b_1 = 1.5b_0$ and for at least four years when $b_1 = 2b_0$. With $v/u = 100$, the optimal consumption in the simulations run is never negative for $b_1 = 1.5b_0$, and is negative in less than 1%

Figure 2

Median of Distribution of Optimal Investment $y^*(t)$ and Mean of Distribution of Optimal Consumption $b^*(t)$ over Time, When v/u Changes

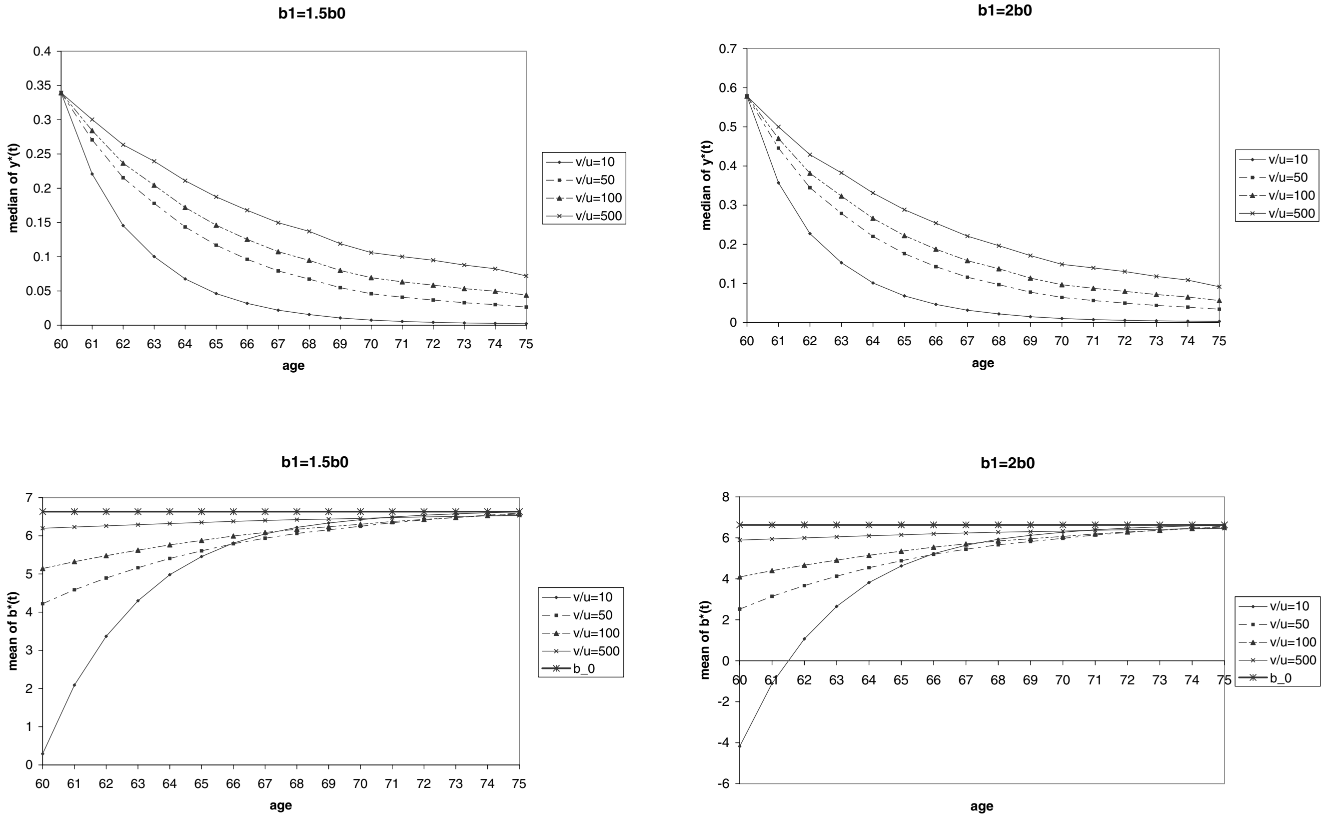
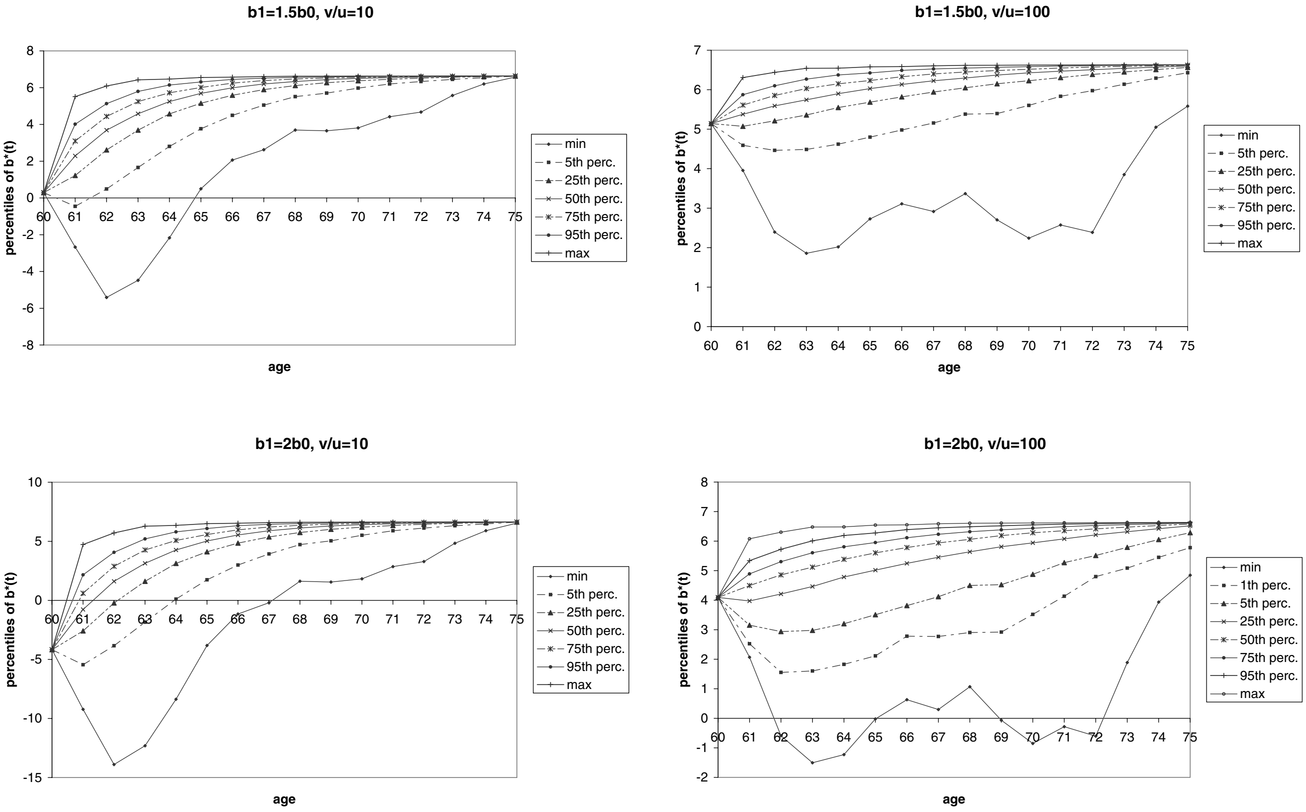


Figure 3

Percentiles of Distribution of Optimal Consumption over Time with *v/u* = 10, 100



of the cases for $b_1 = 2b_0$ (noting that in the last graph also the first percentile of the distribution of the consumption has been plotted).

We notice that the situation of negative consumption may be undesirable to many pensioners, as well as the absence of consumption stability, which makes it diffcult to plan for the future. The main reason for this pattern is the incentive to postpone consumption, due to the relatively high weight attached to the final annuity in comparison with the interim consumption, together with the fact that the desired annuity b_1 is larger than the targeted consumption b_0. However, if the pensioner wishes to avoid negative consumption and needs a more stable consumption path, he or she will choose parameters so as to guarantee it (see Section 4.5).

4.3 Simulation Results: Probability of Ruin, Negative Consumption, Borrowing Money from the Bank, and Ability to Purchase a Better Annuity than the Initial One

Table 2 reports the frequency over the 1,000 simulations of the undesirable events: ruin, negative consumption, and borrowing money from the bank. It reports also the average age when the undesirable event occurs for the first time and the average number of weeks in which the undesirable event occurs (given that it has occurred). The mean and standard deviation of the final annuity are also reported. The ability to purchase a better annuity than the one that was possible to buy at retirement (i.e., b_0) has also been tested. Given that the fund cannot attain the amount needed to buy an annuity of b_1, the aim is to investigate how close to the target b_1 one can get. In particular, the growth of the fund has been monitored at any time between retirement and time T, to see if and when the fund allows the purchase of a certain level of annuity between b_0 and b_1. The four levels $b_{0.5} = b_0 + 0.5(b_1 - b_0)$,

Table 2

Results of Simulations (Without Imposing Restrictions)

Parameter	$b_1 = 1.5b_0 = 9.95$				$b_1 = 2b_0 = 13.26$			
	$\frac{v}{u} = 10$	$\frac{v}{u} = 50$	$\frac{v}{u} = 100$	$\frac{v}{u} = 500$	$\frac{v}{u} = 10$	$\frac{v}{u} = 50$	$\frac{v}{u} = 100$	$\frac{v}{u} = 500$
Ruin probability	0	0	0.2%	1.1%	0	0.1%	0.8%	2.8%
Mean age of ruin	—	—	70	72	—	72	69	69
Prob(neg. cons.)	56.2%	0.2%	0	0	100%	9.7%	1.8%	0
Mean age of neg. cons.	60	63	—	—	60	62	64	—
Mean no. weeks of neg. cons.	21	17	—	—	83	41	22	—
Prob($y^*(t) > 1$)	0	3.1%	7.6%	15.7%	3.5%	21%	28%	37.8%
Mean age of ($y^*(t) > 1$)	—	65	65	66	61	62	62	63
Mean no. weeks ($y^*(t) > 1$)	—	56	71	108	16	54	88	134
Mean of final annuity	9.92	9.63	9.42	9.08	13.22	12.71	12.36	11.78
Standard deviation of final annuity	0.04	0.49	0.81	1.32	0.07	0.84	1.38	2.25
Prob(afford annuity of $b_{0.5}$)	100%	99.1%	96.7%	90.9%	100%	99.2%	97.1%	92.6%
Prob(afford annuity of $b_{0.75}$)	100%	92.7%	84.8%	71.8%	100%	94.3%	87%	75.9%
Prob(afford annuity of $b_{0.9}$)	99.6%	73.2%	54.8%	37.7%	99.8%	76.9%	60.4%	43.1%
Prob(afford annuity of $b_{0.95}$)	98.8%	48.9%	31.5%	17.7%	98.9%	54%	36.6%	21.4%
Mean age when afford ann. of $b_{0.5}$	65	67	68	69	66	68	69	69
Mean age when afford ann. of $b_{0.75}$	70	72	72	73	71	73	73	73
Mean age when afford ann. of $b_{0.9}$	73	74	74	75	74	75	75	75
Mean age when afford ann. of $b_{0.95}$	75	75	75	75	75	75	75	75

$b_{0.75} = b_0 + 0.75(b_1 - b_0)$, $b_{0.9} = b_0 + 0.9(b_1 - b_0)$, and $b_{0.95} = b_0 + 0.95(b_1 - b_0)$ have been tested. The price of the annuity used at any time has been chosen according to the age of the individual at that time, using the mortality table RG48 (males), assuming that the insurance company reviews the annuity price once a year, during the week of the pensioner's birthday. The same interest rate and loading factor adopted for the price of the annuity at age 60 and 75 have been applied.

The frequency of ruin is very low, ranging from 0 to around 1%, and reaching almost 3% only with a high target and a high value of v/u (namely, 500, where consumption is very close to b_0). On average, ruin occurs about 10 years after retirement (when it occurs).

With a low value of v/u (i.e., 10) and a high target ($b_1 = 2b_0$), the optimal consumption is negative immediately after retirement, and on average remains negative for a year and a half; with a a low target ($b_1 = 1.5b_0$) the consumption becomes negative soon after retirement in more than 50% of the cases, and remains negative on average for five months. After that initial period, the consumption turns positive and starts approaching the desired level b_0. On the other hand, the affordability of levels of annuity very close to the (unreachable) target before time T is almost guaranteed: the frequency of achievement of an annuity paying b_α for the rest of the life, with both values of the final target b_1, is about 99–100% for all values of α chosen: 0.5, 0.75, 0.9, and 0.95. It is clear that the initial sacrifice in terms of reduced consumption is compensated by a very high chance of getting as close as one wants to the desired target later. This feature seems important, as there may be the opportunity for the pensioner to renounce a few years of consumption at the beginning of the decumulation phase in order to be able to achieve the desired annuity almost with certainty.

With high enough v/u (i.e., 100), the optimal consumption almost always remains positive. However, the frequency of being able to afford an annuity paying b_α before T is no longer close to 100% (apart from $\alpha = 0.5$) and sharply decreases when α increases, reducing to 55–60% for $\alpha = 0.9$ and dropping to values as low as about 30–35% with $\alpha = 0.95$.

With $v/u = 500$, the probability of reaching the level $b_{0.95}$ before time T goes down to about 20% with both targets. The price that one has to pay for a stable consumption path very close to the desired level b_0 is a lower chance of being able to approach the final annuity target during the drawdown phase, or (which is equivalent) accepting a lower level of lifetime annuity at the time of annuitization. Interestingly, we notice that the chances of getting very close to the target ($\alpha = 0.9$ and 0.95) are slightly higher with $b_1 = 2b_0$ than with $b_1 = 1.5b_0$: this seems to suggest that the higher the target, the higher the reward in terms of chances of approaching it. The same feature is observed in Gerrard, Haberman, and Vigna (2004).

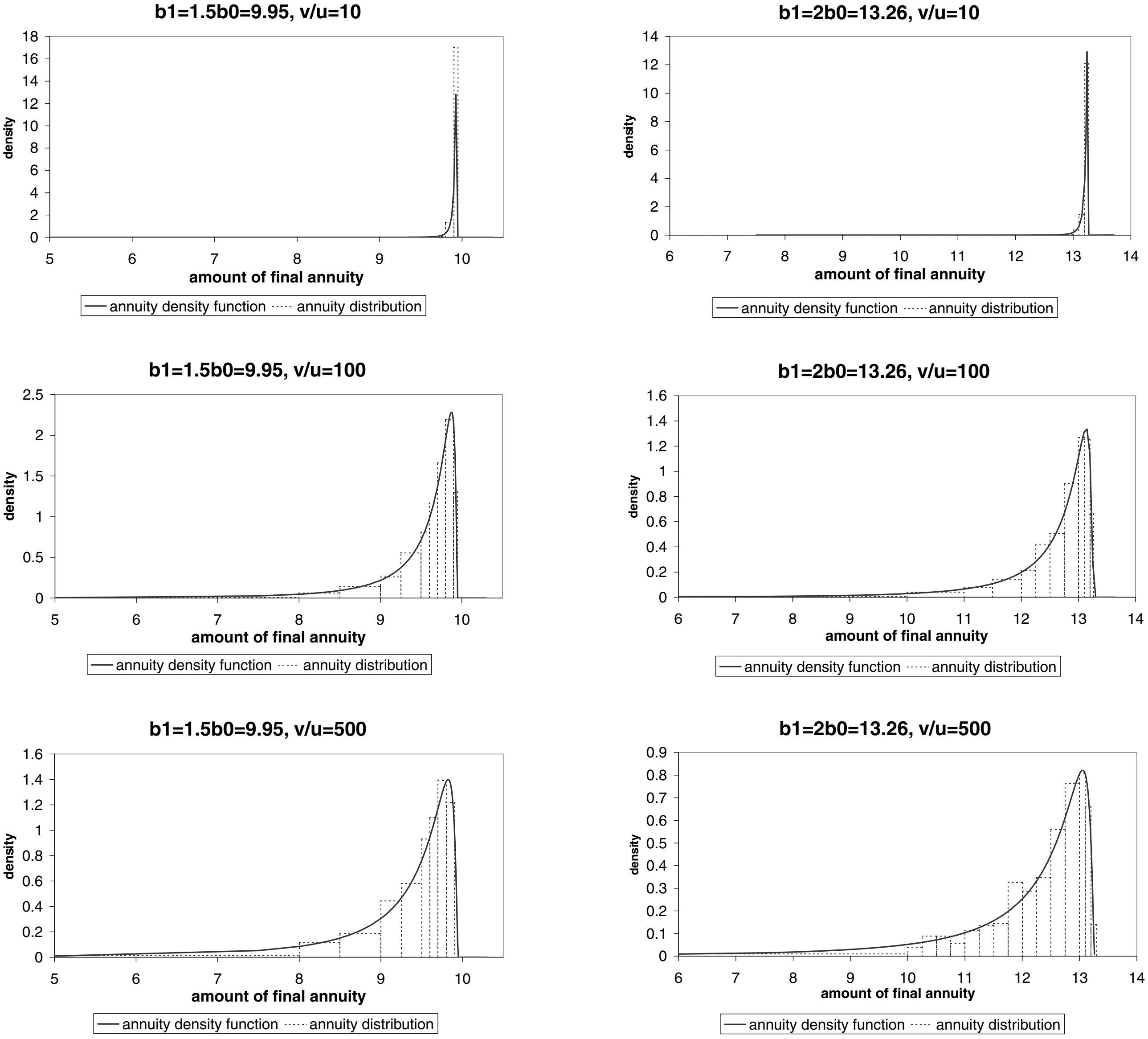
The mean of the final annuity decreases and the standard deviation increases when v/u increases, which is intuitive. Furthermore, the restriction on the investment allocation is violated more often when v/u is raised, with increases both in the probability of borrowing money from the bank to invest in the risky asset and in the number of weeks during which this happens.

4.4 Simulation Results: Density Function and Empirical Distribution of the Final Annuity

The density function of the final annuity $kX(T)$ that can be bought at the time of compulsory annuitization can be exactly calculated. In Figure 4 the density function is plotted together with the histo-

Figure 4

Density Function and Empirical Distribution of Final Annuity



b1=1.5b0=9.95, v/u=10

b1=2b0=13.26, v/u=10

b1=1.5b0=9.95, v/u=100

b1=2b0=13.26, v/u=100

b1=1.5b0=9.95, v/u=500

b1=2b0=13.26, v/u=500

gram of the empirical distribution of the final annuity from the 1,000 simulations, for the two targets and for $v/u = 10$, 100 and 500.

We note that the distribution depends heavily on the choice of the Sharpe ratio, $(\lambda - r)/\sigma$, of the risky asset (for a detailed explanation of the dependence of results on the Sharpe ratio, and for a sensitivity analysis, see Gerrard, Haberman, and Vigna [2004]). The results here reported are relative to a Sharpe ratio of 0.3; if this value were increased (reduced), the distribution would in consequence become more (less) concentrated to the left of the target b_1.

The graphs confirm the results found in Table 2: the higher is the weight given to the running consumption (value of v/u), the more spread out is the distribution of the final annuity.

4.5 Suggestions for Implementation of the Model

This section deals with the difficult but relevant problem of implementing the model in practice. This issue has been tackled by Young (2004): she provides the reader with simple rules for choosing the target rate of consumption by controlling the probability of lifetime ruin. The way that the consumption rate is chosen depends on the maximum tolerable probability of ruin over a specified horizon and other similar issues. In our model, easy rules cannot be provided, simply because we do not have closed formulae for the key measures that are of interest to a pensioner. However, what is available to the decision maker is the probability distribution of the relevant outcomes, derived by the application of the model for specific choices of the parameters. We may imagine that the financial advisors can provide the retirees with tables like Table 2. Based on the quantitative information provided, retirees can understand the likely consequences of their choice and can be helped to make their choices about the setting of the model parameters. For instance, let us consider three different examples, and let us imagine that the table provided by the advisor is Table 2:

1. The pensioner is not solely dependent on the income from the fund, wishes to attain a high target eventually, and is not worried by the possibility of low consumption in the intervening period. This pensioner might be willing to choose the profile $v/u = 10$, $b_1 = 2b_0$, which gives a very high probability of getting close to the desired high annuity.
2. The pensioner needs a regular income for his or her daily needs and has no other sources of income than the one given by the fund. This pensioner probably will select the profile $v/u = 500$ and $b_1 = 1.5b_0$, which gives a consumption path very close to b_0 at all ages.
3. The pensioner has no liquidity problem and has a strong bequest motive. This pensioner probably will choose a very high value for b_1 and the profile $v/u = 10$, so as to keep a high value of the fund for the whole distribution phase. Alternatively, he or she may select a specification other than the natural one for the interim targets, so as to be able to choose a high value of n with some effect on the optimal rules.

In other words, we acknowledge that the choice of the different parameters of the model is not easy a priori, but can be driven by the knowledge of the distribution of the outcomes obtained when implementing the model for specific profiles.

4.6 Imposing Restrictions on the Optimal Choices: Suboptimal Policies

As mentioned in Section 3, the problem with constrained controls has not been solved, because of the difficulty of the task. However, it is possible to act in such a way as to avoid unacceptable situations. In fact, the pensioner can set the consumption equal to 0 whenever the optimal policy would imply negative consumption, and can invest the whole portfolio in the risky asset whenever the optimal policy would imply borrowing money from the bank to invest in the risky asset. Furthermore, the process should be stopped if and when the fund hits the barrier 0, that is, when ruin occurs. The choices just described would not be optimal in the classic sense, in that they would not be the exact solution to the optimal control problem with constraints. However, the difference from the true optimal solution is likely to be small; equally important, these restrictions are quite easy to implement.

We have implemented these suboptimal policies with constraints in a small number of cases and investigated the difference from the results of the unrestricted problem. The scenarios chosen are those where imposing such restrictions is likely to have a significant effect on the investment/consumption choices. Therefore, we have selected $b_1 = 2b_0$, $v/u = 10$ (which is the case where negative consumption is most likely to appear), $b_1 = 2b_0$, $v/u = 100$ (which is the case where borrowing money from the bank is most likely to appear) and $b_1 = 2b_0$, $v/u = 500$ (which is the case where ruin is most likely to appear). Table 3 reports the results. For consistent comparisons, the stream of pseudo-random numbers generated is the same as the one used in the previous case.

Imposing restrictions on the controls does not seem to have a significant effect on the results. Clearly, there are no unreasonable situations any longer (negative consumption and borrowing money from the bank). The frequency of ruin decreases in comparison with the unrestricted case, probably because the high values of $y^*(t)$ are truncated to 1. The mean and standard deviation of the final annuity are similar to the case without restrictions (slightly worse, with the former decreasing and the latter increasing). The chances of getting close to the desired annuity level slightly decrease in all cases (but only by 1–2 percentage points, apart from one case, where it decreases by 5%).

The distribution of the final annuity over the 1,000 simulations is plotted in the graphs of Figure 5. The histograms are very similar to the corresponding histograms of the unrestricted case, and could not be distinguished if they were to be plotted on the same graph.

It may be of interest to observe the behavior of the suboptimal controls when the restrictions are applied. The graphs in Figure 6 report the percentiles of the consumption and of the investment allocation in the cases analyzed. For notational convenience, in the graphs we will indicate as $y^*(t)$ its truncated value.

As expected, the paths of the percentiles of the controls are very stable over time and are more stable than the corresponding percentiles of the unrestricted case. For example, from an inspection of the case $v/u = 100$, we can see that in the unrestricted case the curve of the minimum consumption becomes negative many times between ages 62 and 72 (see Figure 3), whereas it lies strictly above 0 at all ages in the restricted case. This apparent contradiction can be explained by a thorough investigation of the single trajectories in the scenarios where the optimal consumption becomes negative: in the unrestricted case the negative consumption at certain points in time is due to very low values of the fund at that time, while in the restricted case the fund takes higher values, because of the fact that the investment allocation is restricted, and this leads to positive consumption. A further comparison of the single trajectories of the fund in the unrestricted and restricted cases (in the same scenarios for market returns) seems to show that the path of the fund is more stable when restrictions are

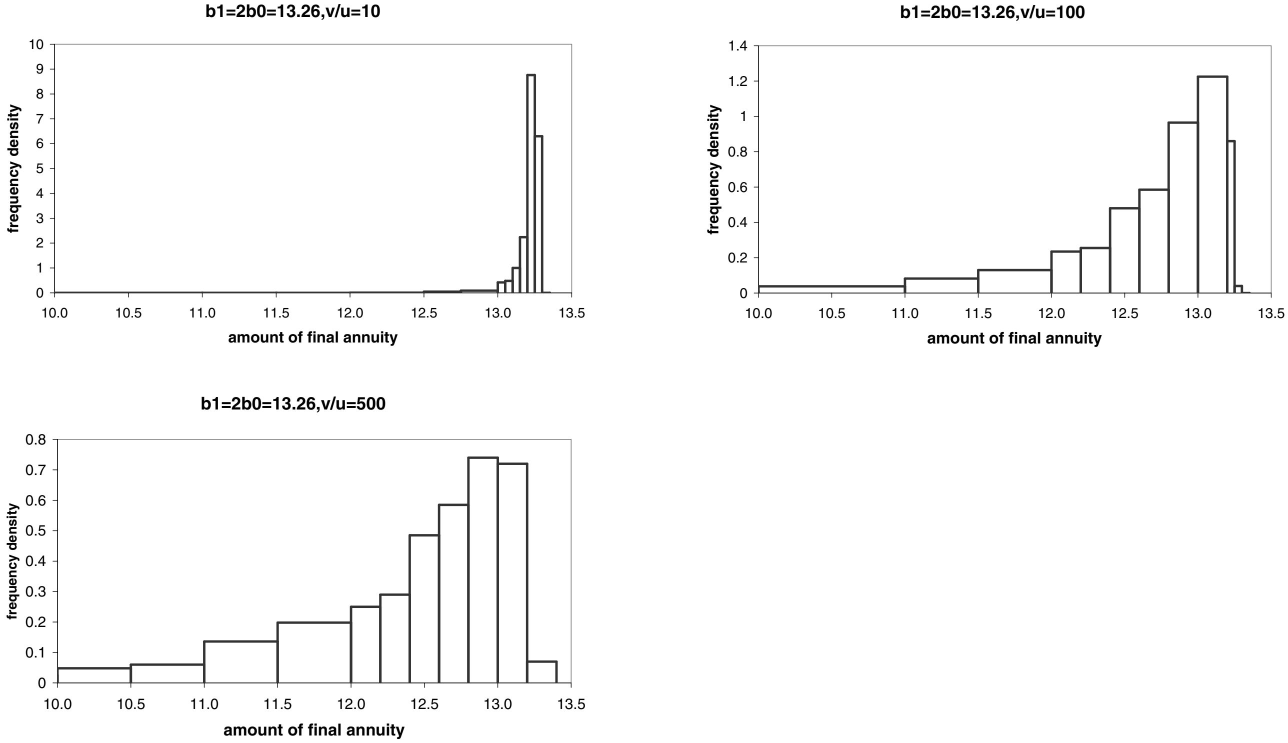
Table 3

Results of Simulations (Imposing Restrictions)

	$b_1 = 2b_0 = 13.26$		
	$\frac{v}{u} = 10$	$\frac{v}{u} = 100$	$\frac{v}{u} = 500$
Ruin probability	0	0	0.4%
Mean age of ruin	—	—	73
Prob(neg. cons.)	0	0	0
Prob($y^*(t) > 1$)	0	0	0
Mean of final annuity	13.19	12.24	11.32
Standard deviation of final annuity	0.29	1.62	2.88
Prob(afford annuity of $b_{0.5}$)	99.9%	95.5%	87.4%
Prob(afford annuity of $b_{0.75}$)	99.8%	85.6%	72.1%
Prob(afford annuity of $b_{0.9}$)	98.9%	59.7%	42.2%
Prob(afford annuity of $b_{0.95}$)	97.4%	36.5%	21%
Mean age when afford annuity of $b_{0.5}$	67	69	69
Mean age when afford annuity of $b_{0.75}$	71	73	73
Mean age when afford annuity of $b_{0.9}$	74	75	75
Mean age when afford annuity of $b_{0.95}$	75	75	75

Figure 5

Distribution of Final Annuity When Imposing Restrictions on Controls via Suboptimal Policies



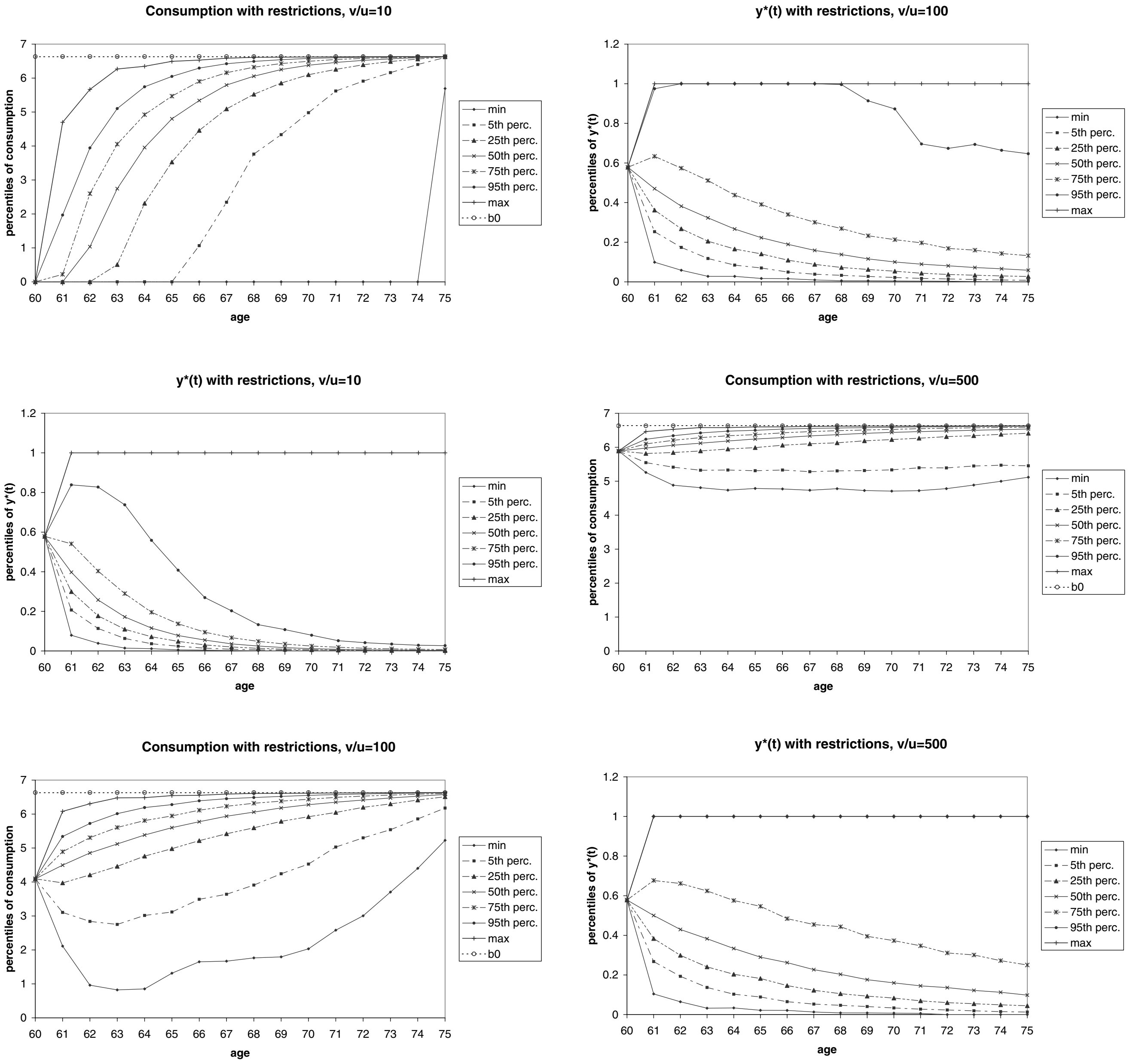
applied, and this, again, is to be explained by the fact that the amount invested in the risky asset cannot exceed the whole portfolio.

5. CONCLUSIONS

In this work, we present an analysis that can lead to a flexible tool that could help the member of a DC pension plan in making his or her decisions in the postretirement phase before annuitization. In particular, he or she can set a desired level of annuity to be bought when ultimate annuitization occurs, and invest and consume in the meantime, according to this target. Mortality has been included in the model, in that the pensioner runs the optimization problem until annuitization or death, whichever occurs first. Furthermore, the individual can give due importance to the ability to leave a bequest in the case of death before annuitization. The problem has been tackled and solved with the techniques of stochastic optimal control theory, in a typical Black and Scholes financial market, with a riskless and a risky asset. We have solved the problem also in the case that consumption is fixed, and the only choice available to the individual is the investment allocation. The solution is found with a particular definition for the target function, which is called the ''natural target,'' in that it acts as a sort of safety level for the needs of the pensioner and takes into account his or her bequest motive and his or her (subjective) force of mortality. With a constant force of mortality, the optimal controls are given in closed form; with a more realistic age-dependent force of mortality, we show that the solution exists and can be well approximated by solving the problem with a constant lower level and a constant higher level for the force of mortality. An unexpected result is that by choosing the natural target, which is linked to the bequest motive, the individual acts optimally as though his or her bequest motive was null. This can be explained by showing that, in general, the bequest motive can be incorporated into the interim target.

Figure 6

Suboptimal Controls When Restrictions Are Applied



In the model presented, the optimal running consumption turns out to be bounded above at a certain level set by the pensioner (which may be useful, in case there are some restrictions on the amount withdrawn periodically from the fund). Similarly, the annuity target chosen can never be reached, but the density function of the final annuity that can be purchased shows that it can be approached very closely.

A sensitivity analysis with respect to the level of the final annuity target and the relative importance given to the level of running consumption and the achievement of the final target has been carried out, by running Monte Carlo simulations for the risky asset. The simulations allow the investigation of relevant issues like the probability of ruin, the occurrence of undesirable or unrealistic events, the probability of being able to buy a better annuity than the one purchasable at retirement, and the distribution of the final annuity that can be bought at the time of compulsory annuitization. By undesirable events we mean negative or too low optimal consumption and optimal investment allocation that implies borrowing money from the bank to invest in the risky asset.

We find that the key characteristics strongly depend on the relative weights given to the level of running consumption during the drawdown phase and to the achievement of the natural target. It can be seen that, by giving high enough importance to the running consumption level, the optimal consumption lies very close to the upper level throughout the drawdown phase. The price that the member has to pay is a less close approach to the natural interim and final targets, and a less favorable distribution of the final annuity. On the other hand, if more importance is attached to the achievement of the natural target, the fund approaches the interim and final targets very closely (a result confirmed by the density function of the final annuity), but the optimal consumption is very likely to be negative at the beginning of the drawdown phase and for a short period of time (up to one to two years) thereafter, approaching the desired level only later on. Therefore, the main conclusion seems to be that the tradeoff between the realization of the different desires of the pensioner regarding consumption and final annuity target can be easily dealt with by choosing appropriate weights for these factors in the initial setting of the optimization problem.

The problem has been solved without restrictions on the optimal investment and consumption choices, because of the difficulty inherent in solving the optimal control problem with constraints. However, we have implemented suboptimal policies by restricting the controls to reasonable boundaries in the simulations, and we have compared the results between the unrestricted and the restricted cases. As expected, the controls applied and the evolution of the fund turn out to be more stable in the restricted case. In addition, the probability of ruin decreases when applying restrictions, and the distribution of the final annuity seems to be very similar to the one obtained in the unrestricted case.

In further research, it would be of interest to solve the problem with constraints numerically and make the comparison with the restricted suboptimal policies applied in this work. Another interesting task would be to investigate the optimal time of annuitization between retirement and the compulsory annuitization age; this is an optimal stopping problem and is the subject of ongoing research.

APPENDIX A

Here we consider the investment allocation problem in the case that consumption is fixed over time and equal to b_0 per unit time. Gerrard, Haberman, and Vigna (2004) consider the same problem in the absence of mortality and the bequest motive. Not surprisingly, this situation turns out to be a natural extension of that above-mentioned work. By letting the weight v given to monitoring of the consumption tend to $+\infty$, we see that it is also a special case of the problem presented in the previous sections.

The growth of the fund is governed by the stochastic differential equation

$$dX(t) = [X(t)(y(t)(\lambda - r) + r) - b_0]dt + X(t)y(t)\sigma dW(t).$$

The running loss function monitors the growth of the fund only:

$$L(t, x) = e^{-\rho t}[u(F(t) - x)^2].$$

The terminal cost in the case of survival at age T is

$$K(x) = we^{-\rho T}(b_1 - kx)^2,$$

and the utility of bequeathing assets of x on death at age t is

$$M(t, x) = e^{-\rho t}nx.$$

The aim is to find the optimal investment allocation to minimize expected future losses, that is, find the optimal value function $V(t, x)$:

$$V(t, x) = \min_{y(\cdot)} H_{t,x}(y(\cdot))$$

with

$$H_{t,x}(y(\cdot)) = \mathbb{E}\left[\int_t^{T\wedge T_D} L(s, X(s))\, ds + K(X(T))1_{T_D>T} - M(T_D, X(T_D))1_{T_D<T}\middle| X(t) = x\right]$$

and find the optimal control $y^*(t)$ such that

$$V(t, x) = H_{t,x}(y^*(t)).$$

The HJB equation is now

$$0 = \min_y \{e^{-\rho t}u(F(t) - x)^2 + V_x[-(b_0 - rx) + (\lambda - r)xy] + V_t + \tfrac{1}{2}\sigma^2x^2y^2V_{xx} - \delta(t)V - \delta(t)e^{-\rho t}nx\}$$

with the same boundary condition as in the problem (2.9).

By trying a value function of the same form as before (see eq. (2.12)), we obtain the following system of differential equations for $A(t)$, $B(t)$, and $C(t)$:

$$\begin{cases} A'(t) = (\rho - 2r + \beta^2 + \delta(t))\, A(t) - u, \\ B'(t) = (\rho + \beta^2 - r + \delta(t))\, B(t) + 2A(t)b_0 + 2uF(t) + n\delta(t), \\ C'(t) = (\rho + \delta(t))C(t) + \dfrac{\beta^2B(t)^2}{4A(t)} + b_0B(t) - uF(t)^2 \end{cases}$$

with the same boundary conditions as in equation (2.14).

REMARK

We notice that the system above is a particular case of the previous system of differential equations (2.13), when the weight v goes to infinity. Furthermore, it is an extension of the system of differential equations solved in Gerrard, Haberman, and Vigna (2004), by adding the terms involving the force of mortality and the importance given to the bequest motive. We also notice that the difficulty inherent in solving the differential equation for $A(t)$ has disappeared, in that we have now a linear differential equation and not a Riccati one.

We proceed to solve it in the same way as before, by introducing the function $G(t)$ and the shortfall $S(t)$. The evolution of the shortfall is now

$$dS(t) = (r - \beta^2)S(t)\, dt - \beta S(t)\, dW(t) + \frac{2u(G(t) - F(t)) - n\delta(t)}{2A(t)}\, dt.$$

If the natural targets are chosen, that is, if equation (3.4) holds, then the shortfall is always positive, and we end up with the same solution as before for the functions $G(t)$ and $F(t)$ (i.e., eqs. (3.5)), which is not so surprising, since the parameter v was not involved in those expressions. What is probably more interesting to notice is that the optimal control $y^*(t)$, which takes the same form as before, is also the same control substantially when the natural targets are chosen:

$$y^*(t, X^*(t)) = \frac{\lambda - r}{\sigma^2}\left(\frac{G(t) - X^*(t)}{X^*(t)}\right).$$

Obviously, the difference will be given by the path of the optimal fund $X^*(t)$, which follows a different SDE, in which consumption is fixed.

Again, the optimal amount invested in the risky asset is a proportion of the shortfall $S(t)$.

APPENDIX B

Before proving Proposition 1, we need to prove Lemma 2:

PROOF OF LEMMA 2

If there is any t such that $A(t) = A_0(t)$, then $A'(t) - A_0'(t) = (\phi(t) - \phi_0)A_0(t)$, and we know that $A_0(t) > 0$. If $\phi(t) - \phi_0$ is always positive, then the only such occurrences involve A crossing A_0 from below, whereas if $\phi(t) - \phi_0$ is negative, then A crosses A_0 from above. In both cases, therefore, there can never be more than one such crossing. But such a crossing occurs at $t = T$, and so that is the only one.

PROOF OF PROPOSITION 1

Let $\phi_U = \max_{t_0 \le t \le T} \phi(t)$, $\phi_L = \min_{t_0 \le t \le T} \phi(t)$, and let $A_U(t), A_L(t)$ be respectively the solutions to equation (2.16) in the case where $\phi \equiv \phi_U$ and the case where $\phi \equiv \phi_L$. Since ϕ is constant in both cases, we may deduce that $A_L(t), A_U(t) > 0$ for all t. As a result of the lemma, $A(t)$ remains sandwiched between the two solutions $A_L(t)$ and $A_U(t)$, and therefore is bounded away from 0.

APPENDIX C

The graphs in Figure A show how the ratio v/u can affect the level of the barrier for negative consumption.

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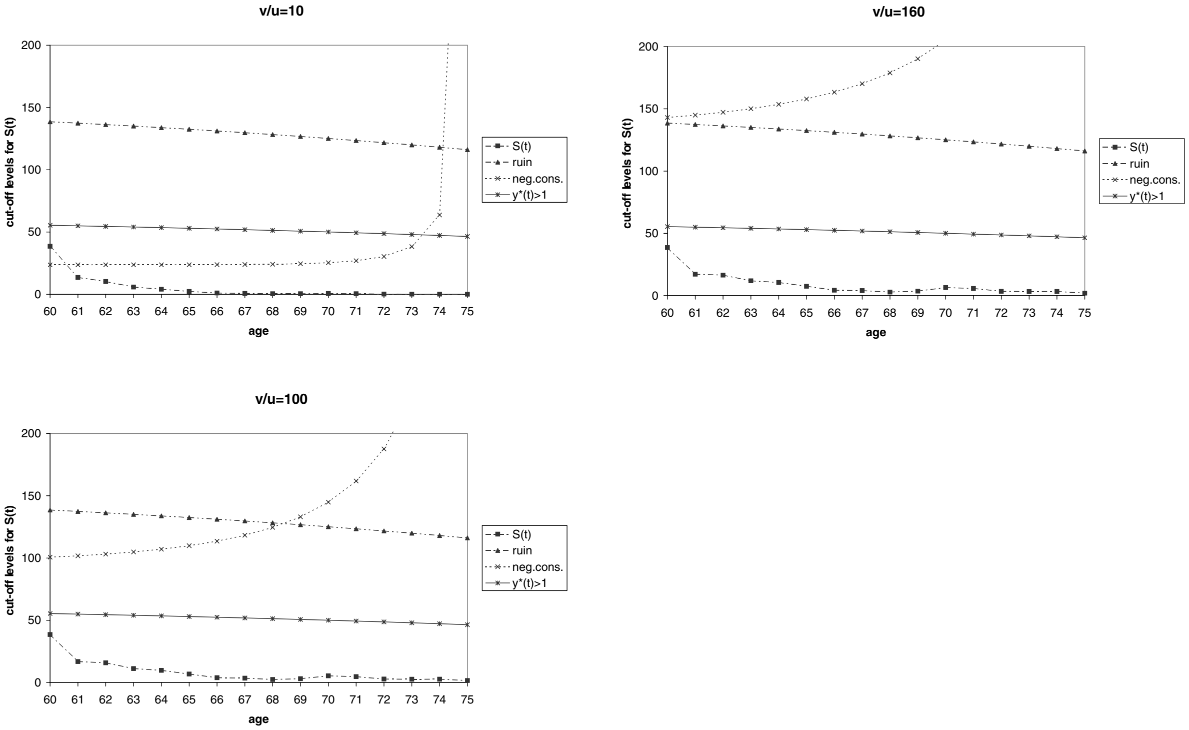
Figure A

Barriers for Shortfall *S*(*t*) for Ruin, Negative Consumption, and Borrowing Money from the Bank, with Different Values of the Ratio *v*/*u*



Note: The path of *S*(*t*) in one particular scenario of market returns is also reported.

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