ON THE EXPECTED DISCOUNTED PENALTY FUNCTION FOR LÉVY RISK PROCESSES

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ABSTRACT

Dufresne et al. (1991) introduced a general risk model defined as the limit of compound Poisson processes. Such a model is either a compound Poisson process itself or a process with an infinite number of small jumps. Later, in a series of now classical papers, the joint distribution of the time of ruin, the surplus before ruin, and the deficit at ruin was studied (Gerber and Shiu 1997, 1998a, 1998b; Gerber and Landry 1998). These works use the classical and the perturbed risk models and hint that the results can be extended to gamma and inverse Gaussian risk processes.

In this paper we work out this extension to a generalized risk model driven by a nondecreasing Lévy process. Unlike the classical case that models the individual claim size distribution and obtains from it the aggregate claims distribution, here the aggregate claims distribution is known in closed form. It is simply the one-dimensional distribution of a subordinator. Embedded in this wide family of risk models we find the gamma, inverse Gaussian, and generalized inverse Gaussian processes. Expressions for the Gerber-Shiu function are given in some of these special cases, and numerical illustrations are provided.

1. INTRODUCTION

We first discuss a risk model with a surplus process of the form

$$U(t) = u + ct - S(t) + \eta Z(t), \qquad t \geq 0, \tag{1.1}$$

where S is a subordinator with Lévy measure dQ and Z is a Lévy motion with no positive jumps and zero drift. Here u is the initial surplus, and c is a constant premium rate defined as $c = (1 + \theta)\,\mathbb{E}[S(1)]$, where $\theta \geq 0$ is the security loading factor. For an account on the classical risk model we refer the reader to Grandell (1991), Bowers et al. (1997), or Asmussen (2000).

Ruin probabilities in this model have been discussed in Bertoin and Doney (1994), Yang and Zhang (2001), Morales and Schoutens (2003), Huzak et al. (2004), Klüppelberg, Kyprianou, and Maller (2004) and Doney and Kyprianou (2006). The results on ruin probabilities for (1.1) follow from well-known results in fluctuation theory for Lévy processes. Huzak et al. (2004) study a ladder-height decomposition for the ruin probability; Klüppelberg, Kyprianou, and Maller (2004) and Doney and Kyprianou (2006) study the ruin probability asymptotics, as well as the over- and undershoot for such a model. However, the Gerber-Shiu function has not yet been explored in such a context.

The Gerber-Shiu function was introduced in Gerber and Shiu (1998a) and was defined to capture several quantities of interest in risk theory, namely, the ruin probability, the Laplace transform of the time to ruin, and the joint density of the surplus prior to ruin and the deficit at ruin. Results for this function are embedded in the quintuple law discussed in Doney and Kyprianou (2006). It is of independent interest to derive and interpret these results in such a way that they remain compatible with

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the existing actuarial literature. It is also useful to explore examples for which computations can be carried out beyond the general expressions.

Dufresne, Gerber, and Shiu (1991) introduced a general risk model defined as the limit of compound Poisson processes. Such a model is either a compound Poisson process itself or a process with an infinite number of small jumps. The construction of Dufresne, Gerber, and Shiu is based on a nonnegative, nonincreasing function that governs the jumps of the process. This function, it turns out, is the tail of the Lévy measure of the process. In this article we work out the extension of work by Dufresne, Gerber, and Shiu for subordinators, a subclass of Lévy processes.

Unlike the classical case, which models the individual claim size distribution and obtains from it the aggregate distribution, here the aggregate claims distribution is known in closed form. In particular, Sections 3.2, 4.3, and 5 discuss examples for which the general results can be carried forward in more detail. This is a straightforward application of the characterization of subordinators, as limits of compound Poisson processes. Nonetheless, it brings new insight to the applicability of Gerber-Shiu functions in general risk models.

We start by defining subordinators and reviewing their first properties.

2. SUBORDINATORS

Subordinators form the subclass of nondecreasing Lévy processes. Lévy processes are in a one-to-one correspondence with the class of infinitely divisible distributions (see Barndorff-Nielsen, Mikosh, and Resnick [2001], Bertoin [1996], or Sato [1999] for accounts on Lévy processes and subordinators). Their characteristic function $\Phi_t(u) = \mathbb{E}(e^{iuX(t)})$ is written in the form $e^{-t\varphi(u)}$, where φ is the so-called characteristic exponent in the Lévy-Khintchine characterization, as given in the following definition.

DEFINITION 2.1

An adapted càdlàg $\mathbb{R}$-valued process $X = \{X(t)_{t\geq 0}$ with $X(0) = 0$ is a Lévy process if its characteristic function is of the form $\Phi_t(u) = e^{-t\varphi(u)}$, where

$$\varphi(u) = -iau + \frac{b^2}{2}u^2 - \int_{\mathbb{R}_0} [e^{iux} - 1 - iux\, \mathbb{I}_{(-1,1)}(x)]\, dQ(x), \qquad u \in \mathbb{R}, \tag{2.1}$$

with $a, b \in \mathbb{R}$, and dQ is a positive measure on $\mathbb{R}_0 = \mathbb{R} - \{0\}$ satisfying

$$\int_{\mathbb{R}_0} (1 \wedge |x|^2)\, dQ(x) < \infty.$$

The constants a and b^2 and the measure dQ uniquely determine X. They are referred to as the triplet of Lévy characteristics (or Lévy triplet for short) $[a, b^2, dQ]$. The measure dQ is called the Lévy measure, and the exponent φ is called the characteristic exponent of the process X.

The Lévy measure governs the occurrence and the size of the jumps of the process X. For instance, the number of jumps of size larger than any $\varepsilon > 0$ is a Poisson process with mean $\bar{Q}(\varepsilon) = \int_\varepsilon^\infty dQ(x)$, where $\bar{Q}$ is the integrated tail of dQ. The density function of the jump sizes is then $dQ(x)/\bar{Q}(\varepsilon)\, \mathbb{I}_{(\varepsilon,\infty)}(x)$. Hence, the jumps larger than ε form a compound Poisson process.

If $b^2 > 0$ and the Lévy measure is identically zero, then the process is a Brownian motion (the only continuous Lévy process). When the Gaussian coefficient $b^2 = 0$ the process is entirely composed by jumps; if in addition $\int_{\mathbb{R}_0} dQ(x) < \infty$, then the process is a compound Poisson process, while if $\int_{\mathbb{R}_0} dQ(x) = \infty$ but $\int_{\mathbb{R}_0} (1 \wedge |x|)\, dQ(x) < \infty$, then the process has an infinite number of small jumps but is of finite variation. Finally, if $\int_{\mathbb{R}_0} dQ(x) = \infty$ and $\int_{R_0} (1 \wedge |x|)\, dQ(x) = \infty$, the process has infinitely many jumps and is of unbounded variation.

If the Gaussian coefficient $b^2 = 0$ and the Lévy measure dQ is defined on $(0, \infty)$, such that $\int_0^\infty (1 \wedge x)\, dQ(x) < \infty$, then the corresponding Lévy process is called a subordinator. Its increments are always positive, and hence its Laplace transform $\phi_t(s) = \mathbb{E}(e^{-sX(t)})$ exists and is of the form $e^{-t\Psi(s)}$, where Ψ is called the Laplace exponent, which can be written as

$$\Psi(s) = as + \int_0^\infty (1 - e^{-sx})\, dQ(x), \qquad s > 0. \tag{2.2}$$

Note the relation with the characteristic exponent in (2.1), $\Psi(s) = \varphi(is)$, for any subordinator. Equation (2.2) characterizes the family of all subordinators. Alternatively, in terms of the integrated tail $\bar{Q}$ of the Lévy measure, we can rewrite (2.2) as (see Bertoin 1996):

$$\frac{\Psi(s)}{s} = a + \int_0^\infty e^{-sx}\bar{Q}(x)\, dx, \qquad s > 0. \tag{2.3}$$

It is useful to define also the cumulant exponent of a Lévy process ϑ. This is simply the exponent appearing in the moment-generating function of the process, that is, $\mathbb{E}(e^{sX(t)}) = e^{-t\vartheta(s)}$. The cumulant exponent is clearly given by the Laplace exponent as $\vartheta(s) = \Psi(-s)$. Note that our definition of ϑ differs in sign from the standard one. This is for consistency with the result of Zolotarev (1964) used later in Theorem 3.1.

2.1 Examples of Subordinators

Illustrative examples of subordinators are the α-stable subordinator, the gamma process, and the generalized inverse Gaussian process.

2.1.1 α-Stable Subordinator

If the Laplace exponent given by

$$\Psi(s) = s^\alpha = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty (1 - e^{-sx})\, x^{-1-\alpha}\, dx, \qquad s > 0, \tag{2.4}$$

with $\alpha \in (0, 1)$, then the process X is called an α-stable subordinator. The Lévy measure is given by

$$dQ(x) = q(x)\, dx = \frac{\alpha}{\Gamma(1-\alpha)} x^{-1-\alpha}\, dx, \qquad x > 0.$$

Notice that $\bar{Q}(0) = \int_0^\infty dQ(x) = \infty$, and therefore the process has infinitely many small jumps.

The α-stable subordinator is a subclass of the larger family of α-stable processes. The restriction on the parameter $0 < \alpha < 1$ is due to the condition $\int_0^\infty (1 \wedge x)\, dQ(x) < \infty$. The increments of this process follow a positive α-stable distribution.

The α-stable family is studied extensively in Janicki and Weron (1994). In insurance the α-stable process has been used recently for risk models in the presence of large claims (Furrer, Michna, and Weron 1997; Furrer 1998). When $\alpha \in [1, 2)$, the α-stable process is no longer a subordinator, but it can be used as the Lévy perturbation in (1.1) (see Section 3).

2.1.2 Gamma Process

The gamma process is a subordinator with Laplace exponent given by

$$\Psi(s) = a \ln\left(1 + \frac{s}{b}\right) = \int_0^\infty (1 - e^{-sx})\, ax^{-1}e^{-bx}\, dx, \qquad s > 0, \tag{2.5}$$

where $a, b > 0$. Clearly the Lévy measure is given by

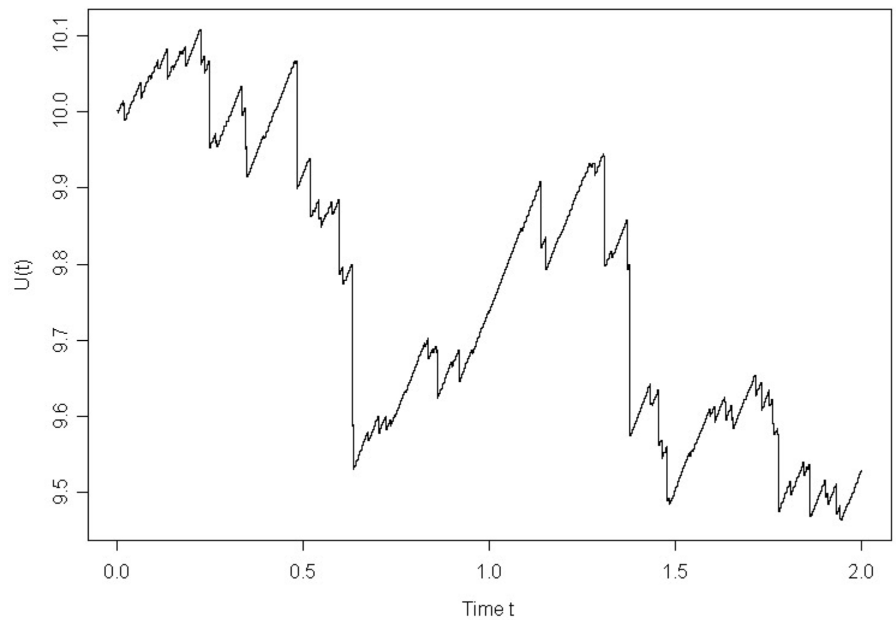
$$dQ(x) = q(x)\, dx = ax^{-1}e^{-bx}\, dx, \qquad x > 0. \tag{2.6}$$

We can easily see that the mean of this process at time 1 is

$$\mu_X = \mathbb{E}[X(1)] = a/b. \tag{2.7}$$

Notice that $\bar{Q}(0) = \int_0^\infty dQ(x) = \infty$, and therefore this process also has infinitely many small jumps. The increments of this process follow a gamma distribution; see the illustrative path in Figure 1 with initial

Figure 1
Surplus Process in (1.1) with Gamma Subordinator (*a* = 15, *b* = 10)



surplus $u = 10$, premium rate $c = 1.3$, gamma parameters $a = 15$ and $b = 10$, but no Lévy perturbation ($\eta = 0$).

This process has been used in finance as well as in insurance (see Dufresne, Gerber, and Shiu 1991; Madan, Carr, and Chang 1998).

2.1.3 Generalized Inverse Gaussian Process

The class of generalized inverse Gaussian (GIG) distributions is characterized by three parameters: β, γ, and κ. If $\beta = -1/2$, the GIG distribution reduces to an inverse Gaussian. The gamma distribution is a limiting case of the GIG distribution for $\beta > 0$, $\gamma > 0$, and $\kappa \to 0$. These make the GIG Lévy processes a natural extention to the gamma process.

GIG distributions have been extensively studied by Jørgensen (1982). Barndorff-Nielsen and Halgreen (1977) showed that this family is infinitely divisible, and therefore we can define a positive Lévy process. The generalized inverse Gaussian process is a subordinator with Laplace exponent given by

$$\Psi(s) = -\ln\left[\frac{K_\beta\left(\kappa\gamma\sqrt{1 + \frac{2s}{\gamma^2}}\right)}{K_\beta(\kappa\gamma)\left(1 + \frac{2s}{\gamma^2}\right)^{\beta/2}}\right] = \int_0^\infty (1 - e^{-sx})\, dQ(x), \tag{2.8}$$

where $\beta \in \mathbb{R}$, κ, $\gamma > 0$, and K_β is the modified Bessel function of the third kind with index β. Its domain is $s > -\gamma^2/2$ when $\beta \geq 0$, but $s \geq -\gamma^2/2$ when $\beta < 0$. Its Lévy measure q is given by

$$dQ(x) = q(x)\, dx = \frac{1}{x}\left[\kappa^2 \int_0^\infty e^{-xt} g_\beta(2\kappa^2 t)\, dt + \max\{0, \beta\}\right] e^{-\gamma^2 x/2}\, dx, \tag{2.9}$$

where

$$g_\beta(y) = \left\{ \frac{\pi^2}{2} y [J^2_{|\beta|}(\sqrt{y}) + N^2_{|\beta|}(\sqrt{y})] \right\}^{-1}.$$

J and N are modified Bessel functions.

The mean of the process at time 1 is

$$\mu_X = \mathbb{E}[X(1)] = \frac{\kappa}{\gamma} \frac{K_{1+\beta}(\kappa\gamma)}{K_\beta(\kappa\gamma)}. \tag{2.10}$$

Note that $\bar{Q}(0) = \int_0^\infty dQ(x) = \infty$, and the process is composed again of an infinite number of small jumps. The increments of length larger than 1 of this process follow a generalized inverse Gaussian distribution.

A particular member of this family, the inverse Gaussian process, has been proven to be a good model for aggregate claims (Chaubey, Garrido, and Trudeau 1998); see the illustrative path in Figure 2 with initial surplus $u = 10$, premium rate $c = 2$, and inverse Gaussian parameters $\beta = -1/2$, $\kappa = 4$, and $\gamma = 2$ (no Lévy perturbation, $\eta = 0$).

A GIG family of processes has been recently proposed in insurance to model aggregate claims (see Morales 2004).

3. A General Perturbed Risk Model

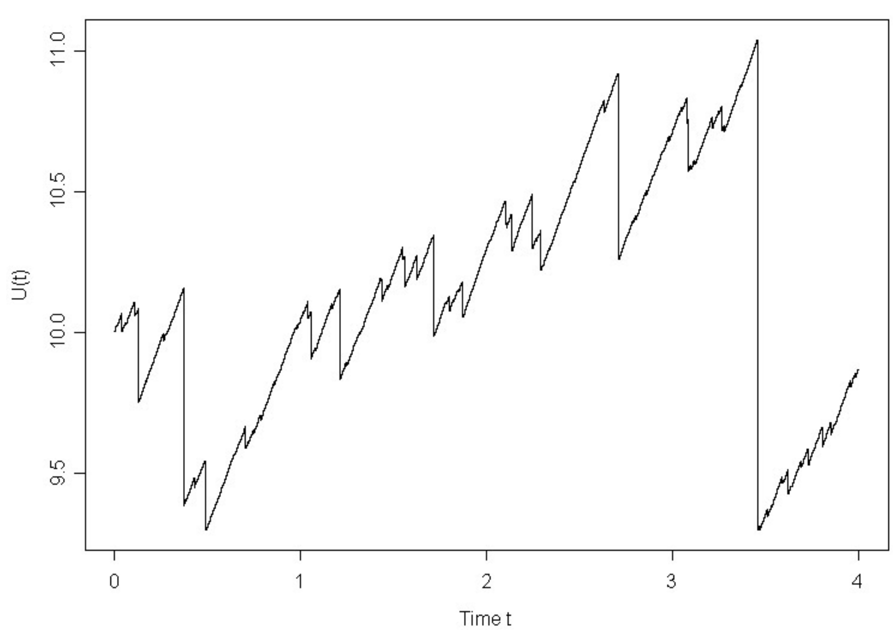
3.1 The Model

Consider a general perturbed risk model based on a subordinator for the aggregate claims and a spectrally negative Lévy process for the perturbation, as in (1.1). This is quite a general model as it includes, as particular cases, several models previously considered in the literature.

Subordinators have appealing features as models for aggregate claims. They are nondecreasing jump processes that can naturally account for claims. Despite their counterintuitive property of allowing for an infinite number of small jumps, they still preserve the ladder-height structure of the classical risk model.

Figure 2

Surplus Process in (1.1) with Inverse Gaussian Subordinator ($\beta = -1/2$, $\kappa = 4$, $\gamma = 2$)



Using a subordinator in the model implies an infinitely divisible distribution for the aggregate claims. Some possible choices are gamma, inverse Gaussian, or generalized inverse Gaussian. All these have closed-form densities and have been used previously as models for aggregate claims. This can be seen as an advantage over defining the individual claims distribution first; recall that in the classical case the aggregate claims distribution does not have a nice closed-form expression.

Despite the loss of detail in subordinator models, we can recover an individual claim size distribution. Simply divide the infinite number of jumps into two types: those smaller than a certain threshold ε, which we will associate with an extra source of randomness (perturbation), and those larger than ε that can be seen as claims with distribution $\bar{Q}(\varepsilon) - \bar{Q}(x)/\bar{Q}(\varepsilon) = \int_\varepsilon^x dQ_\varepsilon(s)/\int_\varepsilon^\infty dQ_\varepsilon(s)$. The tail behavior of such laws depends on the tail behavior of the Lévy measure dQ (see Klüppelberg, Kyprianou, and Maller 2004).

As for the perturbation, spectrally negative Lévy processes are relatively tractable, despite their infinite number of small jumps. Incorporating this type of Lévy process in the model accounts for extra sources of randomness, still keeping a similar but more general structure for the ladder-height decomposition.

This general perturbed model has been recently studied in the literature. Yang and Zhang (2001) proposes a model like (1.1) where Z is a Brownian motion special case. They adapt existing results for Lévy processes to an insurance context. Using a theorem from Zolotarev (1964), they interpret the ascending ladder-height process as a classical ladder-height structure for the ruin probability.

Huzak et al. (2004) work with the general model (1.1) and give a general ladder-height decomposition of the ruin probability. This illustrates the fact that classical results in ruin theory are embedded in the fluctuation theory for spectrally negative Lévy processes.

Chiu and Yin (2005) study the last time that the risk process crosses level zero, as well as the duration of the ruin event. They also use a theorem from Zolotarev (1964).

Both papers above rely on the following result (reproduced here from Zolotarev [1964] for completeness).

Theorem 3.1

Let $X = \{X(t)\}_{t\geq 0}$ be a Lévy process with no positive jumps, cumulant exponent ϑ_X and Lévy measure dQ. Moreover, let it have a finite mean $\mathbb{E}[X(1)] = \mu_X = \int_0^\infty x\, dQ(x) \geq 0$. Define $\psi(u) = \mathbb{P}[\sup_{t\geq 0} - X(t) > u]$ for $u \geq 0$. Then we have that

$$s\int_0^\infty e^{-su}\psi(u)\, du = 1 - \frac{\mu_X s}{\vartheta_X(s)}, \qquad s > 0. \tag{3.1}$$

Above, $\psi(u)$ is associated with the ruin probability for the model in (1.1). Let $X(t) = U(t) - u$, then ψ is the ruin probability for (1.1). Since X has no positive jumps, Theorem 3.1 applies, and a general ladder-height decomposition for ψ can be derived. This is spelled out in the following theorem. (Note: the convolution operator $*$ between two distribution functions F and G on $\mathbb{R}^+$ is defined in the usual way $F * G(x) = \int_{0-}^{x+} F(x - y)\, dG(y) = \int_0^x F(x - y)g(y)\, dy$, if G admits a density function g.)

Theorem 3.2 (Huzak et al. 2004)

Let U be a risk process as in (1.1) and denote by $X(t) = U(t) - u$. Then its associated ruin probability $\psi(u) = \mathbb{P}[\sup_{t\geq 0}\{-X(t)\} > u]$ satisfies the equation

$$1 - \psi(u) = \frac{\theta}{1+\theta}\sum_{n=0}^{\infty}\left(\frac{1}{1+\theta}\right)^n M^{*n} * G^{*(n+1)}(u), \tag{3.2}$$

where M is like a ladder-height distribution, with Laplace transform given by

$$\xi_M(s) = \int_0^\infty e^{-sx}\, dM(x) = \frac{\Psi_S(s)}{\mu_S s}, \tag{3.3}$$

for $\mu_S = \mathbb{E}[S(1)]$ and G is the distribution with Laplace transform given by

$$\xi_G(s) = \int_0^\infty e^{-sx}\, dG(x) = \frac{cs}{\Psi_{ct+\eta Z}(s)}. \tag{3.4}$$

Recall that $\bar{Q}$ *is the tail of the Lévy measure of the subordinator in* (1.1) *and* $\Psi_{ct+\eta Z}$ *is the Laplace exponent of the process* $U(t) - u + S(t) = ct + \eta Z(t)$.

We can recognize ξ_M to be the Laplace transform of a distribution given in terms of the tail of the Lévy measure of the subordinator S, as in (2.3), but with $a = 0$:

$$\xi_M(s) = \frac{\Psi_S(s)}{\mu_S s} = \int_0^\infty e^{-sx} \frac{\bar{Q}(x)}{\int_0^\infty \bar{Q}(t)\, dt}\, dx = \int_0^\infty e^{-sx} \frac{\bar{Q}(x)}{\mu_S}\, dx, \qquad s > 0,$$

which allows us to identify the ladder-height-like density as

$$m(x) = \frac{\bar{Q}(x)}{\int_0^\infty \bar{Q}(t)\, dt}. \tag{3.5}$$

Equation (3.2) implies that $1 - \psi(u)$ follows a defective renewal equation of the form

$$1 - \psi(u) = \frac{\theta}{1+\theta} G(u) + \frac{1}{1+\theta} \int_0^u [1 - \psi(u-s)]\, d(M * G)(s), \qquad u \geq 0. \tag{3.6}$$

From (3.2), using standard renewal theory techniques (see, e.g., Asmussen 2000), we obtain (3.6).

Clearly this unifying approach to risk modeling through Lévy processes not only brings new insight to some well-known models, but also enlarges the class of risk processes. These two effects are illustrated in the following sections.

For instance, Theorem 3.1 gives a relationship between the ruin probability and the Laplace exponent of the risk process. This is used in Section 5.4 to calculate ruin probabilities by inverting Laplace transforms.

Theorem 3.2 gives a general decomposition for the ruin probability that expands the usual ladder-height interpretation. Here each step in the ladder can be decomposed into two independent random variables. This clearly separates the effect of the subordinator from that of the perturbation in each ladder step. The corresponding distributions of these random variables, M and G, are given in terms of their Laplace transforms and are related to the Laplace exponents of the subordinator and the perturbation. This can be used in direct computation to obtain ruin probabilities in a larger class of processes. For instance, the methods described in Politis and Pitts (2005) could be adapted to this more general model.

3.2 Examples

The following illustrative examples will help understand the generality of the model in (1.1).

3.2.1 Classical Compound Poisson Model

The classical risk model is included in (1.1), when S is a compound Poisson process and there is no perturbation, that is,

$$U(t) = u + ct - S(t), \qquad t \geq 0,$$

where the jump distribution is, say, F. We can see that equation (3.6) reduces to the known renewal equation for the ruin probability while its solution (3.2) is the well-known Beekman's convolution formula (see Asmussen 2000, p. 62).

Since there is no perturbation here, we also see that the Laplace transform ξ_G is constant at 1, hence recovering the Laplace transform of a compound geometric. The jumps in such a compound random variable are given by (3.5), which reduces to the so-called ladder-height transformation of the claim distribution F.

3.2.2 α-Stable Perturbation

Furrer (1998) studies the model in (1.1) for a compound Poisson process S and $Z = Z_\alpha$ an α-stable Lévy process with no positive jumps, that is,

$$U(t) = u + ct - S(t) + \eta Z_\alpha(t), \qquad t \geq 0.$$

The compound Poisson process has a jump distribution F. Furrer shows that the function ξ_G of the decomposition in (3.2) is the Laplace transform of a distribution function G given by

$$G(x) = 1 - \sum_{n=0}^{\infty} \frac{(-c/\eta^\alpha)^n}{\Gamma(1 + (\alpha - 1)n)} x^{(\alpha-1)n}, \qquad x \geq 0. \tag{3.7}$$

Hence the renewal equation (3.6) and its solution (3.2) hold with the above G, with the ladder-height transformation M of the jump distribution F:

$$M(x) = \frac{1}{\beta} \int_0^x [1 - F(t)]\, dt, \qquad x > 0, \tag{3.8}$$

and where $\beta = \int_0^\infty [1 - F(x)]\, dx < \infty$ is the mean jump.

3.2.3 Brownian Motion Perturbation

Gerber (1972) and Dufresne and Gerber (1991) study model (1.1) when $Z = W$ is a Brownian motion with zero drift and infinitesimal variance σ^2, and S is a compound Poisson process, that is,

$$U(t) = u + ct - S(t) + \sigma W(t), \qquad t \geq 0.$$

This is a particular case of the model of Furrer (1998), since a Brownian motion is an α-stable Lévy process with index $\alpha = 2$. Equations (3.2) and (3.6) hold with M given by (3.8). As for the function G in (3.7), it reduces to

$$G(x) = 1 - e^{-(c/\sigma^2)x}, \qquad x > 0,$$

that is, an exponential distribution.

3.2.4 Gamma

Dufresne, Gerber, and Shiu (1991) study model (1.1) when S is a gamma process and the perturbation Z is not present, that is,

$$U(t) = u + ct - S(t), \qquad t \geq 0.$$

If the aggregate process is a gamma process, then equation (3.2) takes the form

$$1 - \psi(u) = \frac{\theta}{1 + \theta} \sum_{n=0}^{\infty} \left(\frac{1}{1 + \theta}\right)^n M^{*n}(u), \qquad u \geq 0, \tag{3.9}$$

where M is the distribution function of m in (3.5), that is,

$$M(x) = \frac{\int_0^x \bar{Q}(t)\, dt}{\int_0^\infty \bar{Q}(t)\, dt} \tag{3.10}$$

$$= b \int_0^x E_1(bt)\, dt = 1 - e^{-bx} bxE_1(bx), \qquad x \geq 0, \tag{3.11}$$

and where $E_1(x) = \int_x^\infty t^{-1}e^{-t}\, dt$ is the exponential integral function (see Abramowitz and Stegun 1970).

This last equation comes from the fact that, for a gamma process with Lévy measure given by (2.6), we have

$$\int_0^\infty \bar{Q}(t)\, dt = \int_0^\infty t\, dQ(t) = \frac{a}{b},$$

while

$$\int_0^x \bar{Q}(t)\, dt = \int_0^x \int_t^\infty as^{-1}e^{-bs}\, ds\, dt = a \int_0^x E_1(bt)\, dt, \qquad \text{for } x \geq 0.$$

Notice that (3.9) implies that ψ satisfies the renewal equation in (3.6):

$$\psi(u) = \frac{1 - M(u)}{1 + \theta} + \frac{1}{1 + \theta} \int_0^u \psi(u - y)\, dM(y), \qquad u \geq 0. \tag{3.12}$$

In other words, (3.6) simplifies here to

$$\psi(u) = \frac{e^{-bu} - buE_1(bu)}{1 + \theta} + \frac{1}{1 + \theta} \int_0^u \psi(u - y) bE_1(by)\, dy, \qquad u \geq 0, \tag{3.13}$$

where E_1 is as in (3.11). Note that both (3.12) and (3.13) imply $\psi(0) = 1/(1 + \theta)$, as in the classical case (see also Yang and Zhang 2001, p. 288).

3.2.5 Generalized Inverse Gaussian

In Morales (2004) we find a model like (1.1) where S is a generalized inverse Gaussian Lévy process, without the perturbation Z:

$$U(t) = u + ct - S(t), \qquad t \geq 0.$$

In this case (3.2) takes the form

$$1 - \psi(u) = \frac{\theta}{1 + \theta} \sum_{n=0}^{\infty} \left(\frac{1}{1 + \theta}\right)^n M^{*n}(u), \qquad u \geq 0, \tag{3.14}$$

where M is as in (3.10) and $\bar{Q}$ is the tail of the Lévy measure of a GIG process, given in (2.9). This expression must be computed numerically. Note again that $\psi(0) = 1/(1 + \theta)$ here also, as in the above gamma case.

However, if the parameter β of the GIG is $\pm\frac{1}{2}$, then closed forms for M exist since the function g_β in (2.9) becomes $g_\beta(y) = 1/\pi\sqrt{y}$. In particular, if $\beta = 1/2$, M becomes

$$M(x) = 1 - \frac{\gamma^2}{\kappa\gamma + 1} \left\{ \frac{2\kappa\Gamma(\frac{1}{2})}{\pi\gamma} \bar{\Gamma}\left(\frac{\gamma^2}{2} x; \frac{1}{2}\right) - \frac{\kappa\Gamma(\frac{1}{2})}{\pi} \left[\frac{2}{\gamma} \bar{\Gamma}\left(\frac{\gamma^2}{2} x; \frac{3}{2}\right) - \gamma x \bar{\Gamma}\left(\frac{\gamma^2}{2} x; \frac{1}{2}\right)\right] + \frac{1}{2}\left[\frac{2}{\gamma^2} e^{-(\gamma 2/2)x} - x\bar{\Gamma}\left(\frac{\gamma^2}{2} x; 0\right)\right]\right\}, \tag{3.15}$$

where $\bar{\Gamma}(u; \alpha) = \int_u^\infty x^{\alpha-1}e^{-x}\, dx$ is the tail of the usual incomplete gamma function (see Abramowitz and Stegun 1970, p. 260). Some authors differ by defining $\bar{\Gamma}(u; \alpha) = 1/\Gamma(\alpha) \int_u^\infty x^{\alpha-1}e^{-x}\, dx$). Note that our $\bar{\Gamma}(u; 0)$ is also the exponential integral function $-E_1(-u)$ in (3.11).

If $\beta = -1/2$ (inverse Gaussian process), then M is given by

$$M(x) = 1 - \frac{\gamma}{\kappa} \left\{ \frac{2\kappa\Gamma(\frac{1}{2})}{\pi\gamma} \bar{\Gamma}\left(\frac{\gamma^2}{2} x; \frac{1}{2}\right) - \frac{\kappa\Gamma(\frac{1}{2})}{\pi} \left[\frac{2}{\gamma} \bar{\Gamma}\left(\frac{\gamma^2}{2} x; \frac{3}{2}\right) - \gamma x \bar{\Gamma}\left(\frac{\gamma^2}{2} x; \frac{1}{2}\right)\right]\right\}. \tag{3.16}$$

This last expression comes from the fact that, for a GIG process with Lévy measure given by (2.9), we have

$$\int_0^\infty \bar{Q}(t)\, dt = \int_0^\infty t\, dQ(t) = \frac{\kappa\gamma + 1}{\gamma^2}.$$

Note that (3.14) implies that ψ satisfies a renewal equation in terms of the function M. For instance, in the case where $\beta = \frac{1}{2}$, equation (3.12) can be written in terms of M, in (3.15), or its corresponding density function m given for $x \geq 0$ by (see Morales 2004 for details):

$$m(x) = \frac{\gamma^2}{[\gamma\kappa + 1]} \left[\frac{\kappa\Gamma(\frac{1}{2})}{\pi\sqrt{2}} \int_x^\infty t^{-3/2} e^{-(\gamma^2/2t)}\, dt + \frac{1}{2} \int_x^\infty t^{-1} e^{-(\gamma^2/2t)}\, dt \right]. \tag{3.17}$$

4. EXPECTED DISCOUNTED PENALTY FUNCTION

Gerber and Shiu (1998a) introduces the concept of expected discounted penalty function as a way to study the distribution of the time to ruin and the surplus at and prior to ruin. For the classical risk model, the expected discounted penalty function ϕ, henceforth called the Gerber-Shiu (G-S) function, is defined as follows:

$$\phi(u) = \mathbb{E}[w(U(\tau-), |U(\tau)|) e^{-\delta\tau} \mathbb{I}_{[0,\infty)}(\tau) | U(0) = u], \qquad u \geq 0, \tag{4.1}$$

where $w(x, y)$ is a nonnegative penalty function for being ruined, u is the initial surplus, τ is the time of ruin, and $U(\tau-)$ is the surplus just prior to ruin.

Gerber and Shiu (1998a) show that (4.1) can be written as a convolution series:

$$\phi(u) = h \star \sum_{k=0}^{\infty} g^{\star k}(u), \qquad u \geq 0, \tag{4.2}$$

for some functions h and g (where $h \star g = \int_0^x h(x - y) g(y)\, dy$, which differs from the convolution of distributions, $F \star G$, defined for Theorem 3.2 and denoted by a different asterisk).

The G-S function can be extended to more general risk processes like model (1.1). This problem will be worked out in its full generality in a sequel to this paper. We first study (1.1) without the perturbation Z, that is, $\eta = 0$, and hence

$$U(t) = u + ct - S(t), \qquad t \geq 0, \tag{4.3}$$

where S is a subordinator.

Appendix A.1 shows that, for the risk process driven by a subordinator S only ($\eta = 0$), the G-S function in (4.1) satisfies (4.2) with

$$g(x) = \frac{1}{1 + \theta} \int_x^\infty e^{-\rho(y-x)} \frac{q(y)}{\int_0^\infty \bar{Q}(t)\, dt}\, dy, \qquad x \geq 0 \tag{4.4}$$

and

$$h(x) = \frac{1}{1 + \theta} \int_x^\infty \int_0^\infty e^{-\rho(z-x)} w(z, y) \frac{q(z + y)}{\int_0^\infty \bar{Q}(t)\, dt}\, dy\, dz, \qquad x \geq 0. \tag{4.5}$$

Here $dQ(y) = q(y)\, dy$ and $\bar{Q}$ are, respectively, the Lévy measure of the subordinator S in (4.3) and its corresponding integrated tail. As for the constant ρ, it is the non-negative solution of the equation

$$\delta - cr + \Psi(r) = 0, \qquad r \geq 0, \tag{4.6}$$

where Ψ is the Laplace exponent of the subordinator in (4.3).

REMARK 4.1

Note that equation (4.2) *implies that the G-S function is the solution of the equation*

$$\phi(z) = \int_0^z \phi(x) g(z - x)\, dx + h(z), \qquad z > 0, \tag{4.7}$$

where the functions g and h are as in (4.4) *and* (4.5), *respectively.*

REMARK 4.2

Denote by $\hat{f}$ the Laplace transform $\hat{f}(s) = \int_0^\infty e^{-sx} f(x)\, dx$ of a function f, for $s \geq 0$. Then the solution ϕ of (4.7) can be expressed also in terms of its Laplace transform $\hat{\phi}$:

$$\hat{\phi}(s) = \hat{\phi}(s)\hat{g}(s) + \hat{h}(s) = \sum_{k=0}^{\infty} [\hat{g}(s)]^k \hat{h}(s) = \frac{\hat{h}(s)}{1 - \hat{g}(s)}, \qquad s \geq 0.$$

For subordinators, the structure of the Gerber-Shiu function is preserved and related results follow in the same straightforward way. For example, if $\delta = 0$, then $\rho = 0$ and the differential term

$$g(y)\, dy = \frac{1}{1+\theta} \frac{\bar{Q}(y)}{\int_0^\infty \bar{Q}(x)\, dx}\, dy = \frac{1}{1+\theta}\, m(y)\, dy$$

can be interpreted as the probability that the surplus will ever fall below u and be between $u - y$ and $u - y - dy$. Recall that m is the ladder-height density for the subordinator S.

Also, if $\delta = 0$ and $w(x, y) = 1$, then

$$h(x) = \frac{1}{1+\theta} \int_x^\infty m(t)\, dt = \frac{1}{1+\theta} \bar{M}(x),$$

which can be interpreted as the probability that the surplus will ever fall below u and will be below $u - x$ when it happens for the first time.

Other results that can be obtained by approaching risk models from the theory of Lévy processes. For example, a relation between the G-S function ϕ for a risk process driven by a subordinator and the ruin probability function ψ is derived in the following result (see Appendix A.2 for the proof).

Proposition 4.1

Let ϕ be the expected discounted penalty function as defined in (4.1) for a risk process driven by a subordinator S with Lévy measure dQ. Let also h be the function defined in (4.5). Then the Laplace transform of ϕ is given by the product

$$\hat{\phi}(u) = \frac{1+\theta_\rho}{\theta_\rho} \hat{h}(u)\hat{\eta}(u), \qquad u \geq 0, \tag{4.8}$$

where

$$\hat{\eta}(s) = \frac{\dfrac{\theta_\rho}{1+\theta_\rho}}{1 - \dfrac{1}{1+\theta_\rho} \hat{\psi}_{\bar{M}_\rho}(s - \rho)}$$

is the Laplace transform of the tail of a compound geometric distribution. Moreover, this distribution is the ruin probability ψ_ρ of a risk process driven by a subordinator S_ρ (the $(-\rho)$-Esscher transform of the original subordinator S) and loading factor: $\theta_\rho = (1 + \theta)\mu_S/\mu_\rho - 1$, where $\mu_\rho = \mathbb{E}[S_\rho(1)]$.

This representation of the Gerber-Shiu function in terms of ruin probabilities of an Esscher-transformed risk process seems to be new. It can be compared to a similar expression derived in Drekic, Stafford, and Willmot (2004) for the classical risk model. Hence (4.8) should extend to subordinators their method to compute moments for the time of ruin.

4.1 Ruin Probabilities

If $w(x, y) = 1$ and $\delta = 0$, then $\rho = 0$ and the G-S function reduces to the ultimate ruin probability, that is, the renewal equation in (4.7) becomes

$$\psi(u) = \frac{1 - M(u)}{1 + \theta} + \frac{1}{1 + \theta} \int_0^u \psi(u - y)\, dM(y), \qquad u \geq 0. \tag{4.9}$$

Recall that m and M are, respectively, the ladder-height density and distribution defined in (3.5) and (3.10). Also, for $w(x, y) = 1$ and $\delta = 0$, (4.4) and (4.5) become, respectively,

$$g(x) = \frac{1}{1 + \theta} \int_x^\infty \frac{q(y)}{\int_0^\infty \bar{Q}(t)\, dt}\, dy = \frac{1}{1 + \theta} \frac{\bar{Q}(x)}{\int_0^\infty \bar{Q}(t)\, dt} = \frac{m(x)}{1 + \theta}, \qquad x \geq 0,$$

and

$$h(x) = \frac{1}{1 + \theta} \int_x^\infty \int_0^\infty \frac{q(z + y)}{\int_0^\infty \bar{Q}(t)\, dt}\, dy\, dz = \int_x^\infty g(t)\, dt = \frac{1 - M(x)}{1 + \theta}, \qquad x \geq 0.$$

Ruin probabilities can be computed numerically using (4.9), as its solution is a convolution series. Alternatively, Section 5 gives examples that use the relation between ruin probabilities and the Laplace exponent in Theorem 3.1. A numerical inversion of the Laplace transform then gives the ruin probabilities.

4.2 Joint Distribution of the Surplus Prior and at Ruin

Consider the discounted density function

$$f_\delta(x, y|u) = \int_0^\infty e^{-\delta t} f(x, y, t|u)\, dt,$$

where $f(x, y, t|u)$ is the joint density of the surplus prior to ruin $U(\tau-)$, the deficit at ruin $|U(\tau)|$, and the time to ruin τ. Note that the discounted distribution F_δ can be recovered from the G-S function when w is an indicator function assigning value 1 to the point (x_0, y_0), that is, $\phi(u) = F_\delta(x_0, y_0|u)$ for $w(x, y) = \mathbb{I}_{[0,x_0)}(x)\, \mathbb{I}_{[0,y_0)}(y)$. The discounted density f_δ is then obtained by differentiation.

Also note that if $\delta = 0$, then $f_\delta(x_0, y_0|u) := f(x_0, y_0|u)$ is simply the joint density of the surplus prior to and the deficit at ruin. This implies that $f_\delta(x, y|u)$, and therefore $f(x, y|u)$, both follow a renewal equation of the form in (4.7), that is,

$$f_\delta(x, y|u) = \int_0^u f_\delta(x, y|z) g(u - z)\, dz + h(u), \qquad u > 0, \tag{4.10}$$

where g is given in (4.4), and h in (4.5) simplifies to

$$h(z) = \frac{1}{1 + \theta} e^{-\rho(x - z)} \frac{q(x + y)}{\int_0^\infty \bar{Q}(t)\, dt} \mathbb{I}_{[0,x)}(z), \qquad z \geq 0.$$

Hence, the solution of (4.10) is

$$f_\delta(x, y|u) = h \star \sum_{k=0}^\infty g^{\star k}(u), \qquad u \geq 0.$$

Classical results for the discounted joint density $f_\delta(x, y|0)$ and its marginals also can be extended to the model (4.3) in a straightforward way.

From (4.10) we have that $\phi(0) = h(0)$, and hence

$$f_\delta(x, y|0) = \frac{1}{1 + \theta} e^{-\rho x} \frac{q(x + y)}{\int_0^\infty \bar{Q}(t)\, dt}, \qquad x > 0, y > 0.$$

Integrating this function over x gives the discounted marginal of the deficit at ruin:

$$f_{\delta,2}(y|0) = \int_0^\infty f_\delta(x, y|0)\, dx = g(y). \tag{4.11}$$

Similarly, the discounted marginal of the surplus before ruin is given by

$$f_{\delta,1}(x|0) = \int_0^\infty f_\delta(x, y|0)\, dy = \frac{1}{1+\theta} e^{-\rho x}\, m(x). \tag{4.12}$$

Finally, the Laplace transform of the time to ruin τ is

$$\begin{aligned}\mathbb{E}[e^{-\delta\tau}\, \mathbb{I}_{[0,\infty)}(\tau)|U(0) = 0] &= \int_0^\infty \int_0^\infty f_\delta(x, y|0)\, dy\, dx \\ &= \frac{1}{1+\theta} \int_0^\infty e^{-\rho x}\, m(x)\, dx.\end{aligned}$$

4.3 Examples

The above results are illustrated with the gamma and generalized inverse Gaussian risk models of Section 3.2.

4.3.1 Gamma Process

If the aggregate claims form a gamma process in (4.3), then the G-S function (4.2) is the solution of the renewal equation in (4.7), with the functions g and h in (4.8) and (4.9), respectively. The latter simplify when the Lévy measure of a gamma process in (2.5) is used:

$$g(x) = \frac{1}{1+\theta} \int_x^\infty e^{-\rho(y-x)}\, by^{-1}e^{-by}\, dy, \qquad x \geq 0, \tag{4.13}$$

and for $x \geq 0$,

$$h(x) = \frac{1}{1+\theta} \int_x^\infty \int_0^\infty e^{-\rho(z-x)}\, w(z, y) b(z + y)^{-1} e^{-b(z+y)}\, dy\, dz, \tag{4.14}$$

where ρ is given by (4.6), which is here the nonnegative solution of

$$-\delta + cr + a \ln\left(1 + \frac{r}{b}\right) = 0. \tag{4.15}$$

These simplifications are due to the fact that for the gamma process $\int_0^\infty \bar{Q}(t)\, dt = \int_0^\infty t\, dQ(t) = a/b$.

Setting $w(x, y) = 1$ and $\delta = 0$ to obtain ruin probabilities, yields $\rho = 0$, and equations (4.13) and (4.14) become

$$g(x) = \frac{1}{1+\theta} \int_x^\infty by^{-1}e^{-by}\, dy = \frac{1}{1+\theta} bE_1(bx),$$

and

$$h(x) = \int_x^\infty g(y)\, dy = \frac{e^{-bx} - bxE_1(bx)}{1+\theta},$$

respectively.

The G-S function then gives the same ultimate ruin probability as in (3.13). Note that the resulting renewal equation for the ultimate ruin probability is also consistent with (3.12).

Similarly, the discounted joint density $f_\delta(x, y|u)$ satisfies a renewal equation of the form in (4.10), that is,

$$f_\delta(x, y|u) = \int_0^u f_\delta(x, y|z)g(u - z)\, dz + h(u), \qquad u > 0, \tag{4.16}$$

where g is given by (4.13) and

$$h(z) = \frac{1}{1 + \theta} e^{-\rho(x-z)} b(x + y)^{-1} e^{-b(x+y)} \mathrm{I}_{[0,x)}(z), \qquad z \geq 0.$$

Finally, (4.11) and (4.12) show that for the gamma process

$$f_{\delta,2}(y|0) = \frac{e^{\rho y}}{1 + \theta} \int_y^\infty bx^{-1}\, e^{-(b+\rho)x}\, dx = \frac{be^{\rho y}}{1 + \theta} E_1[(b + \rho)y], \qquad y > 0,$$

while the corresponding discounted marginal of the surplus before ruin is given by

$$f_{\delta,1}(x|0) = \frac{1}{1 + \theta} e^{-\rho x} m(x) = \frac{e^{-\rho x}}{1 + \theta} bE_1(bx), \qquad x > 0.$$

4.3.2 Generalized Inverse Gaussian

If the aggregate claims in (4.3) form a generalized inverse Gaussian process, then the G-S function (4.2) is the solution of the renewal equation (4.7), where the functions g and h are given by (4.4) and (4.5). The latter simplify when the Lévy measure of the GIG process in (2.9) is used. Simpler expressions are obtained when $\beta = \pm\frac{1}{2}$.

For example, if $\beta = \frac{1}{2}$, then (4.4) and (4.5) take these forms, for $x \geq 0$:

$$g(x) = \frac{1}{1 + \theta} \int_x^\infty e^{-\rho(y-x)} \frac{\gamma^2}{(\kappa\gamma + 1)} \left[\frac{\kappa\Gamma(1/2)}{\pi\sqrt{2y}} + \frac{1}{2}\right] y^{-1} e^{-\gamma^2 y/2}\, dy, \tag{4.17}$$

and

$$\begin{aligned} h(x) = \frac{1}{1 + \theta} \int_x^\infty \int_0^\infty e^{-\rho(z-x)} w(z, y) \frac{\gamma^2}{(\kappa\gamma + 1)} \left[\frac{\kappa\Gamma(1/2)}{\pi\sqrt{2(z + y)}} + \frac{1}{2}\right] \\ \times (z + y)^{-1} e^{-\gamma^2(z+y)/2}\, dy\, dz, \qquad x \geq 0, \end{aligned} \tag{4.18}$$

where ρ is given by (4.6) and is here the nonnegative solution of

$$-\delta + cr - \ln\left[\frac{K_\beta\left(\kappa\gamma\sqrt{1 + \dfrac{2r}{\gamma^2}}\right)}{K_\beta(\kappa\gamma)\left(1 + \dfrac{2r}{\gamma^2}\right)^{\beta/2}}\right] = 0. \tag{4.19}$$

These follow from the GIG Lévy measure in (2.9) and $\beta = \frac{1}{2}$, which imply that $\int_0^\infty \bar{Q}(t)\, dt = \int_0^\infty t\, dQ(t) = \kappa\gamma + 1/\gamma^2$ and

$$q(x) = \left[\frac{\kappa}{\pi\sqrt{2x}} \Gamma(1/2) + \frac{1}{2}\right] \frac{1}{x} e^{-\gamma^2 x/2}, \qquad x > 0.$$

See Morales (2004) for details.

To obtain ruin probabilities, set $w(x, y) = 1$ and $\delta = 0$, which yields $\rho = 0$, and the above g and h simplify to

$$g(x) = \frac{1}{1+\theta}\frac{\gamma^2}{(\kappa\gamma+1)}\left[\frac{\kappa\Gamma(\frac{1}{2})}{\pi\sqrt{2t}} + \frac{1}{2}\right]\int_x^\infty t^{-1}e^{-\gamma^2/2t}\,dt \tag{4.20}$$

and

$$\begin{aligned} h(x) &= \frac{1-M(x)}{1+\theta} \\ &= \frac{1}{1+\theta}\frac{\gamma^2}{(\kappa\gamma+1)}\left\{\frac{2\kappa\Gamma(\frac{1}{2})}{\pi\gamma}\bar{\Gamma}\left(\frac{\gamma^2}{2}x;\frac{1}{2}\right) - \frac{\kappa\Gamma(\frac{1}{2})}{\pi}\left[\frac{2}{\gamma}\bar{\Gamma}\left(\frac{\gamma^2}{2}x;\frac{3}{2}\right)\right.\right. \\ &\quad \left.\left. - \gamma x\bar{\Gamma}\left(\frac{\gamma^2}{2}x;\frac{1}{2}\right)\right] + \frac{1}{2}\left[\frac{2}{\gamma^2}e^{-(\gamma^2/2)x} - x\bar{\Gamma}\left(\frac{\gamma^2}{2}x;0\right)\right]\right\}, \qquad x > 0. \end{aligned}$$

The G-S function then reduces to the ultimate ruin probability, and the solution to (4.7) is consistent with the ruin function obtained in Section 3.2.5.

The discounted density $f_\delta(x, y|u)$ of a GIG with $\beta = \frac{1}{2}$ satisfies a renewal equation of the form in (4.10) with g is given by (4.17) and

$$\begin{aligned} h(z) &= \frac{1}{1+\theta}e^{-\rho(x-z)}\frac{\gamma^2}{(\kappa\gamma+1)}\left[\frac{\kappa\Gamma(1/2)}{\pi\sqrt{2(y+x)}} + \frac{1}{2}\right] \\ &\quad \times (x+y)^{-1}e^{-\gamma^2(x+y)/2}\mathbb{1}_{[0,x)}(z), \qquad z \geq 0. \end{aligned}$$

By contrast, for parameter $\beta = -\frac{1}{2}$ (inverse Gaussian), these functions become

$$g(z) = \frac{1}{1+\theta}\int_z^\infty e^{-\rho(y-z)}\frac{\gamma}{\kappa}\left[\frac{\kappa\Gamma(1/2)}{\pi\sqrt{2y}}\right]y^{-1}e^{-\gamma^2 y/2}\,dy, \qquad z \geq 0,$$

and

$$\begin{aligned} h(z) &= \frac{1}{1+\theta}e^{-\rho(x-z)}\frac{\gamma}{\kappa}\left[\frac{\kappa\Gamma(1/2)}{\pi\sqrt{2(x+y)}}\right] \\ &\quad \times (x+y)^{-1}e^{-\gamma^2(x+y)/2}\mathbb{1}_{[0,x)}(z), \qquad z \geq 0. \end{aligned}$$

Finally, (4.11), (4.12), and (4.17) imply that for the GIG process with $\beta = \frac{1}{2}$:

$$\begin{aligned} f_{\delta,2}(y|0) &= \frac{e^{\rho y}}{1+\theta}\frac{\gamma^2}{(\kappa\gamma+1)}\left\{\frac{\sqrt{2}\kappa\Gamma(1/2)}{\pi}\left[y^{-1/2}e^{-(\gamma^2/2+\rho)y} - \sqrt{(\gamma^2/2+\rho)}\right.\right. \\ &\quad \left.\left. \times\bar{\Gamma}\left(\left(\frac{\gamma^2}{2}+\rho\right)y;\frac{1}{2}\right)\right] + \frac{1}{2}E_1\left(\left(\frac{\gamma^2}{2}+\rho\right)y\right)\right\}, \qquad y > 0, \end{aligned}$$

where $\bar{\Gamma}(u;\alpha) = \int_u^\infty x^{\alpha-1}e^{-x}\,dx$ as in (3.15). The corresponding discounted marginal of the surplus before ruin is given from (3.17) as

$$\begin{aligned} f_{\delta,1}(x|0) &= \frac{1}{1+\theta}e^{-\rho x}m(x), \qquad x > 0, \\ &= \frac{e^{-\rho x}}{1+\theta}\frac{\gamma^2}{(\kappa\gamma+1)}\left\{\frac{\sqrt{2}\kappa\Gamma(\frac{1}{2})}{\pi}\left[x^{-1/2}e^{-\gamma^2 x/2} - \frac{\gamma}{\sqrt{2}}\bar{\Gamma}\left(\frac{\gamma^2}{2}x;\frac{1}{2}\right)\right]\right. \\ &\quad \left. + \frac{1}{2}E_1\left(\frac{\gamma^2}{2}y\right)\right\}. \end{aligned}$$

On the other hand, for the GIG with parameter $\beta = -\frac{1}{2}$ (inverse Gaussian) the last two expressions take the following slightly simpler forms:

$$f_{\delta,2}(y|0) = \frac{e^{\rho y}}{1+\theta}\frac{\gamma}{\kappa}\left\{\frac{\sqrt{2}\kappa\Gamma(1/2)}{\pi}\left[y^{-1/2}e^{-(\gamma^2/2+\rho)y} - \sqrt{(\gamma^2/2+\rho)}\right.\right.$$
$$\left.\left.\times\,\bar{\Gamma}\left(\left(\frac{\gamma^2}{2}+\rho\right)y;\frac{1}{2}\right)\right]\right\},\qquad y>0,$$

and, from (3.16),

$$f_{\delta,1}(x|0) = \frac{1}{1+\theta}e^{-\rho x}m(x),\qquad x>0,$$
$$= \frac{e^{-\rho x}}{1+\theta}\frac{\gamma}{\kappa}\left\{\frac{\sqrt{2}\kappa\Gamma(\frac{1}{2})}{\pi}\left[x^{-1/2}e^{-\gamma^2 x/2} - \frac{\gamma}{\sqrt{2}}\bar{\Gamma}\left(\frac{\gamma^2}{2}x;\frac{1}{2}\right)\right]\right\}.$$

Note that in these examples, when $\delta = 0$, then $\rho = 0$ also, and

$$f_{0,1}(x|0) = f_{0,2}(x|0) := f(x|0),\qquad x>0,$$

a consequence of the known duality property

$$f_0(x,y|0) = f_0(y,x|0) = \frac{1}{1+\theta}\frac{q(x+y)}{\int_0^\infty \bar{Q}(t)\,dt},\qquad x,y>0.$$

It is clearly preserved here for risk processes driven by subordinators.

5. NUMERICAL EXAMPLES

This last section illustrates numerically some of the above results. In particular, expression (3.1) in Theorem 3.1 is used to compute ruin probabilities. It relates the Laplace transform of the ruin probability and the Laplace exponent of the aggregate claims process. Ruin probabilities are then obtained by numerical inversion of the Laplace transform for the model

$$U(t) = u + ct - S(t),\qquad t\geq 0, \tag{5.1}$$

when the aggregate claims process S is gamma, inverse Gaussian, and generalized inverse Gaussian process.

5.1 Gamma Process

Using (2.5) and (2.7) in (3.1) yields the Laplace transform of the ruin probability $\hat{\psi}$, the gamma subordinator risk model:

$$\hat{\psi}(s) = \frac{1}{s} - \frac{c - a/b}{cs - a\ln(1+s/b)}. \tag{5.2}$$

Recall that the process values $X(t)$ in Theorem 3.1 refer to $U(t) - u$ in risk model (1.1), and, therefore, $\mu_X = \mathbb{E}[X(1)] = \mathbb{E}[U(1) - u] = c - a/b$.

Using standard numerical inversion techniques on (5.2) produces the ruin probabilities illustrated in Table 1.

5.2 Inverse Gaussian Process

The inverse Gaussian process is a particular case of the GIG when $\beta = -\frac{1}{2}$. Using (2.8) and (2.10), with this choice of β in (3.1), yields the following Laplace transform of the ruin probability:

$$\hat{\psi}(s) = \frac{1}{s} - \frac{c - \kappa/\gamma}{cs + \kappa\gamma[1 - \sqrt{1+2s/\gamma^2}]}. \tag{5.3}$$

Here $\mu_X = \mathbb{E}[X(1)] = \mathbb{E}[U(1) - u] = c - \kappa/\gamma$. We also use the fact that for $\beta = \pm\frac{1}{2}$ the modified Bessel function K_β simplifies to

$$K_{-1/2}(x) = K_{1/2}(x) = \sqrt{\frac{\pi}{2}}\, x^{-1/2}e^{-x}. \tag{5.4}$$

Again, the values in Table 1 were obtained inverting (5.3) numerically.

5.3 GIG Process

The generalized inverse Gaussian process also takes on a simpler form for $\beta = \frac{1}{2}$. Here $\mu_X = \mathbb{E}[X(1)] = \mathbb{E}[U(1) - u] = c - \kappa/\gamma\ [1 + (\kappa\gamma)^{-1}]$. Using (5.4) and the fact that

$$K_{1+1/2}(x) = \sqrt{\frac{\pi}{2}}\, x^{-1/2}e^{-x}[1 + x^{-1}],$$

the Laplace transform of the ruin probability ψ becomes

$$\hat{\psi}(s) = \frac{1}{s} - \frac{c - \dfrac{\kappa}{\gamma}\,[1 + (\kappa\gamma)^{-1}]}{cs + \kappa\gamma[1 - \sqrt{1 + 2s/\gamma^2}] - \ln\,[\sqrt{1 + 2s/\gamma^2}]}. \tag{5.5}$$

Table 1 lists an illustrative example of the GIG process.

5.4 Ruin Probabilities

Table 1 gives the ruin probability values in all three models above for different values of the loading θ and the initial surplus u. For comparison purposes all models illustrated have expected claims equal to two per time unit. The following parameters were used: gamma process ($a = 2.2$, $b = 1.1$), IG process ($\kappa = 1.8$, $\gamma = 0.9$), and GIG process ($\beta = \frac{1}{2}$, $\kappa = 1$, $\gamma = 1$).

For fixed expected aggregate claims, the gamma process is the least risky, while the GIG is the most risky, in terms of ultimate ruin probabilities.

This computational method for ruin probabilities can be used to obtain the figures in Dufresne, Gerber, and Shiu (1991), for the gamma process, and in Morales (2004) for the generalized inverse Gaussian process. Their parameters differ from those used here. Still our method and program reproduce their ruin probability values when the same parameters are used.

Table 1

Ruin Probabilities for Gamma [a = 2.2, b = 1.1], Inverse Gaussian [κ = 1.8, γ = 0.9], and Generalized Inverse Gaussian [β = 1/2, κ = 1, γ = 1]

	θ = 0.2			θ = 0.5		
u	Gamma	IG	GIG	Gamma	IG	GIG
1	0.55525	0.57042	0.61963	0.30334	0.32773	0.37812
2	0.39059	0.44236	0.49991	0.15644	0.20918	0.25466
3	0.27592	0.34798	0.40685	0.08244	0.13846	0.17545
4	0.19548	0.27537	0.33217	0.04405	0.09336	0.12224
5	0.13894	0.21869	0.27167	0.02368	0.06373	0.08584
6	0.09908	0.17413	0.22249	0.01266	0.04389	0.06063
7	0.07086	0.13897	0.18245	0.00660	0.03041	0.04302
8	0.05078	0.11114	0.14980	0.00325	0.02115	0.03061
9	0.03641	0.08904	0.12315	0.00140	0.01473	0.02180
10	0.02608	0.07145	0.10136	0.00039	0.01024	0.01552

6. CONCLUSIONS

We study the Gerber-Shiu function for a risk process driven by a subordinator, deriving tractable analytical expressions in some special cases, like the gamma, inverse Gaussian, and generalized inverse Gaussian processes.

This modeling approach has the advantage of yielding a known distribution for the aggregate claims. It contrasts with the classical approach to risk modeling that specifies the individual claim severity distribution, requiring convolution processes to derive the resulting aggregate claims distribution.

One disadvantage of subordinator risk models is their implied infinite number of small claims. Still, the model remains mathematically tractable, and small jumps can be interpreted as an extra source of variation. This is akin to other models in the literature (Dufresne, Gerber, and Shiu 1991; Furrer 1998; Furrer, Michua, and Weron 1997; Huzak et al. 2004; Klüppelberg, Kyprianou, and Maller 2004; Morales 2004; Morales and Schoutens 2003; Yang and Zhang 2001).

A unifying approach to risk modeling through Lévy processes brings new insight on some well-known risk models. It also enlarges the class of risk processes that allow the computation of several ruin-related quantities.

The fluctuation theory for spectrally negative Lévy processes has evolved in parallel to the study of G-S functions. Although they intersect, one theory is not a subset of the other. For instance, the G-S function is now available for some Sparre Anderson risk models that are not Lévy processes. Similarly, the theory of exit times for Lévy processes considers more general hitting times than the crossing of ruin or dividend barriers.

We provide here a review of classical ruin problems, as seen from a Lévy processes perspective. It gives two unifying computational methods for the ruin probability of any subordinator risk process and reproduces the known ruin probabilities in the gamma and GIG cases. One method is based on the solution of a renewal equation and requires evaluating series of convolutions. The other relies on the numerical inversion of Laplace transforms. Clearly G-S functions can be further explored with these examples to obtain moments for the time of ruin or asymptotic expressions for the probability of ruin or of overshoot.

The G-S function for the general model (1.1) remains an objective for future research. It requires a slight redefinition of the G-S function, as in the perturbed model identifying which jumps represent claims is no longer clear. Another goal for future research is to explore the implications on G-S functions of the latest findings in quintuple laws for Lévy processes.

APPENDIX

A.1. PROOF OF (4.4) AND (4.5)

Let ϕ be the discounted penalty function for the risk process in (4.3). S is a subordinator, and it has an infinite number of small jumps. However, jumps larger than a fixed positive threshold form a compound Poisson process. Therefore we can construct, for any $\varepsilon > 0$, a classical risk processes U_ε as follows:

$$U_\varepsilon(t) = u + c_\varepsilon t - S_\varepsilon(t), \qquad t \geq 0, \tag{A.1}$$

where S_ε is a compound Poisson process with jump density

$$f_\varepsilon(x)\ dx = \frac{dQ(x)}{\bar{Q}(\varepsilon)}\, \mathbb{I}_{(\varepsilon,\infty)}(x),$$

and arrival rate

$$\lambda_\varepsilon = \int_\varepsilon^\infty dQ(t) = \bar{Q}(\varepsilon). \tag{A.2}$$

The loaded premium is given by

$$c_\varepsilon = (1+\theta)\lambda_\varepsilon\mu_\varepsilon = (1+\theta)\bar{Q}(\varepsilon)\int_\varepsilon^\infty x\,\frac{dQ(x)}{\bar{Q}(\varepsilon)}$$
$$= (1+\theta)\left[\varepsilon\bar{Q}(\varepsilon) + \int_\varepsilon^\infty \bar{Q}(x)\,dx\right], \tag{A.3}$$

where μ_ε is the mean of f_ε. As $\varepsilon \to 0$, the family of processes in (A.1) converges weakly to the process in (4.3). This follows from the fact that the U_ε are Lévy processes and their triplets converge to the triplet of the process U in (4.3).

On the other hand, since the process in (A.1) is a classical risk process for any $\varepsilon > 0$, then its G-S function ϕ_ε takes the form

$$\phi_\varepsilon(u) = h_\varepsilon \star \sum_{k=0}^{\infty} g_\varepsilon^{\star k}(u), \qquad u \geq 0, \tag{A.4}$$

where, from (2.31) of Gerber and Shiu (1998a),

$$g_\varepsilon(x) = \frac{\lambda_\varepsilon}{c_\varepsilon}\int_x^\infty e^{-\rho_\varepsilon(y-x)}\,\frac{q(y)}{\bar{Q}(\varepsilon)}\,dy, \qquad x \geq 0, \tag{A.5}$$

and from (2.33) of Gerber and Shiu (1998a),

$$h_\varepsilon(x) = \frac{\lambda_\varepsilon}{c_\varepsilon}\int_x^\infty\int_0^\infty e^{-\rho_\varepsilon(z-x)}w(z, y)\,\frac{q(z+y)}{\bar{Q}(\varepsilon)}\,dy\,dz, \qquad x \geq 0. \tag{A.6}$$

The coefficient ρ_ε is the nonnegative solution of (4.6), which here is

$$\delta - cr + \Psi_\varepsilon(r) = 0, \tag{A.7}$$

where Ψ_ε is the Laplace exponent of the compound Poisson process S_ε in (A.1).

To see that $\phi_\varepsilon \to \phi$ as $\varepsilon \to 0$, first note that (A.7) converges to (4.6). It follows from (A.2) and (A.3) that (A.5) and (A.6) converge to (4.4) and (4.5), respectively. Finally, the G-S function (A.4) then converges to (A.2). □

A.2. PROOF OF PROPOSITION 4.1

To verify (4.8) we first need to define the Esscher transform of a process. Let $X = \{X(t)\}$ be a subordinator with Lévy triplet $(a, 0, dQ)$. Define $X^\varrho = \{X^\varrho(t)\}$, the ϱ-Esscher transform of X, as the process induced by the density process $e^{\varrho X(t)}/\mathbb{E}[e^{\varrho X(t)}]$. Then it can be seen that X^ϱ is a Lévy process with triplet $[a_\varrho = a, 0, e^{\varrho x}\,dQ(x)]$ (see Morales and Schoutens 2003 and Sato 1999 for a proof).

Now, from (4.2) we have that the Laplace transform of ϕ can be written as

$$\hat{\phi}(s) = \hat{h}(s)\sum_{k=0}^{\infty}\hat{g}^k(s) = \frac{\hat{h}(s)}{1-\hat{g}(s)}. \tag{A.8}$$

On the other hand, from (4.4) we have the Laplace transform of g:

$$\hat{g}(s) = \frac{1}{1+\theta}\int_0^\infty e^{-(s-\rho)x}\int_x^\infty e^{-\rho y}\,\frac{q(y)}{\int_0^\infty \bar{Q}(t)\,dt}\,dy\,dx.$$

From above we see that $e^{-\rho y}\,dQ(y)$ is the Lévy measure dQ_ρ of a $(-\rho)$-Esscher-transformed process. Hence $\hat{g}(s)$ can be rewritten as

$$\hat{g}(s) = \frac{1}{1+\theta}\int_0^\infty e^{-(s-\rho)x}\,\frac{\mu_\rho}{\mu_S}\int_x^\infty \frac{q_\rho(y)}{\int_0^\infty \bar{Q}_\rho(t)\,dt}\,dy\,dx, \tag{A.9}$$

where $\mu_\rho = \mathbb{E}[S_\rho(1)] = \int_0^\infty \bar{Q}_\rho(t)\,dt = \int_0^\infty tq_\rho(t)\,dt$. The latter can be rewritten as

$$\mu_\rho = \mu_S + \int_0^\infty x(e^{-\rho x} - 1)\, dQ(x).$$

From (A.9) we can identify $q_\rho(y)/\int_0^\infty \bar{Q}_\rho(t)\, dt$ as the ladder-height density $m_\rho(y)$ of a risk process driven by S_ρ. Therefore, we can write (A.9) as

$$\hat{g}(s) = \frac{1}{1+\theta}\int_0^\infty e^{-(s-\rho)x}\,\frac{\mu_\rho}{\mu_S}\int_x^\infty m_\rho(y)\, dy\, dx = \frac{1}{1+\theta}\,\frac{\mu_\rho}{\mu_S}\,\hat{\xi}_{\bar{M}_\rho}(s-\rho),$$

where $\hat{\xi}_{\bar{M}_\rho}$ is the Laplace transform of $\bar{M}_\rho(x) = \int_x^\infty m_\rho(y)\, dy$. Substituting this last expression into (A.8) gives

$$\hat{\phi}(s) = \frac{\hat{h}(s)}{1 - \dfrac{1}{1+\theta}\,\dfrac{\mu_\rho}{\mu_S}\,\hat{\xi}_{\bar{M}_\rho}(s-\rho)}.$$

Letting $\theta_\rho = (1+\theta)\mu_S/\mu_\rho - 1$, then multiplying and dividing by $\theta_\rho/(\theta_\rho + 1)$ gives (4.8).

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DISCUSSION

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The authors are to be congratulated for this comprehensive paper on modeling an insurer's surplus using stochastic processes with stationary and independent increments and without upward jumps. A main purpose of this discussion is to promote a recipe given by Dufresne, Gerber, and Shiu (1991, p. 185) for translating results in the classical case where $\{S(t)\}$ is a compound Poisson process to the more general case where $\{S(t)\}$ is a subordinator (a process with stationary, independent, and nonnegative increments). Start with a formula for the compound Poisson process model with Poisson parameter λ and individual claim amount distribution $P(y)$ and density $p(y)$. Then substitute $\bar{Q}(y)$ for $\lambda[1 - P(y)]$ or $q(y)$ for $\lambda p(y)$ to obtain the corresponding formula for the subordinator model. (Note that $\bar{Q}(y)$ in the paper corresponds to $Q(y)$ in Dufresne, Gerber, and Shiu [1991].)

Consider formulas (2.30), (2.33), (3.3), and (3.6) in Gerber and Shiu (1998):

$$g(x) = \frac{\lambda}{c}\int_x^\infty e^{-\rho(y-x)}p(y)\,dy, \tag{D.1}$$

$$h(x) = \frac{\lambda}{c}\int_x^\infty \int_0^\infty e^{-\rho(z-x)}w(z, y)p(z + y)\,dy\,dz, \tag{D.2}$$

$$f_\delta(x, y|0) = \frac{\lambda}{c}\,e^{-\rho x}p(x + y), \tag{D.3}$$

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$$\mathbb{E}[e^{-\delta\tau}\mathbb{1}_{[0,\infty)}(\tau) < \infty | U(0) = 0] = \frac{\lambda}{c}\int_0^\infty e^{-\rho x}[1 - P(x)]\,dx. \tag{D.4}$$

Then the recipe translates these formulas to

$$g(x) = \frac{1}{c}\int_x^\infty e^{-\rho(y-x)}q(y)\,dy, \tag{D.5}$$

$$h(x) = \frac{1}{c}\int_x^\infty \int_0^\infty e^{-\rho(z-x)}w(z, y)q(z + y)\,dy\,dz, \tag{D.6}$$

$$f_\delta(x, y|0) = \frac{1}{c}e^{-\rho x}q(x + y), \tag{D.7}$$

$$\mathbb{E}[e^{-\delta\tau}\mathbb{1}_{[0,\infty)}(\tau) < \infty | U(0) = 0] = \frac{1}{c}\int_0^\infty e^{-\rho x}\bar{Q}(x)\,dx. \tag{D.8}$$

Because

$$\int_0^\infty \bar{Q}(y)\,dy = \mathbb{E}[S(1)] = \frac{c}{1 + \theta}, \tag{D.9}$$

formulas (D.5) and (D.6) are, respectively, formulas (4.4) and (4.5) in the paper. Formulas (D.7) and (D.8) can be found in Section 4.2 of the paper.

For the compound Poisson case, *Lundberg's fundamental equation* (Gerber and Shiu 1998, [2.22]) is

$$0 = \delta + \lambda - c\xi - \lambda\hat{p}(\xi) = \delta - c\xi + \lambda\int_0^\infty (1 - e^{-\xi x})p(x)\,dx. \tag{D.10}$$

Its translation is

$$0 = \delta - c\xi + \int_0^\infty (1 - e^{-\xi x})q(x)\,dx = \delta - c\xi + \xi\hat{\bar{Q}}(\xi). \tag{D.11}$$

This is the same as equation (4.6) of the paper, because

$$\Psi(\xi) = \xi\hat{\bar{Q}}(\xi) \tag{D.12}$$

by equation (2.3) with $a = 0$.

It is well known (Bowers et al. 1997, Theorem 13.5.1) that the quantity $(\lambda/c)[1 - P(y)]\ dy$ can be interpreted as the probability that the compound Poisson surplus process will ever drop below its initial value u and will be between $u - y$ and $u - y - dy$ when it happens for the first time. Thus, for the subordinator case, the formula is $(1/c)\bar{Q}(y)\ dy$. By conditioning on the amount of this drop, we obtain the following equation for the probability of eventual ruin $\psi(u)$:

$$\psi(u) = \frac{1}{c}\int_0^u \psi(u - y)\bar{Q}(y)\,dy + \frac{1}{c}\int_u^\infty \bar{Q}(y)\,dy, \qquad u \geq 0. \tag{D.13}$$

This is the same as equations (3.12) and (4.9) in the paper, because

$$M(u) = \frac{\int_0^u \bar{Q}(y)\,dy}{\int_0^\infty \bar{Q}(y)\,dy} = \frac{1 + \theta}{c}\int_0^u \bar{Q}(y)\,dy \tag{D.14}$$

by formula (D.9).

We want to mention that the $(-\rho)$-Esscher transform in Proposition 4.1 reminded us of Chan, Gerber, and Shiu (2006) and Yin (2006); a $(-\rho)$-Esscher transform appears in both discussions. Also, Dickson and Waters (1993) is an elegant paper on finite-time survival probabilities.

We end this discussion with a brief discussion on dividend strategies for the subordinator model. The surplus model is modified in that dividends are paid to the shareholders of the insurance company. We assume that the dividends are paid according to a *barrier strategy* corresponding to a barrier at the level b. Thus, whenever the surplus is on the barrier b, dividends are paid at rate c. If the surplus is below b, no dividends are paid. Evidently ruin will occur with certainty in this model. For $0 \le u \le b$, let $V(u; b)$ denote the expected present value of the dividend payments until ruin, where u is the initial surplus and b is the barrier level. Let h be a positive function whose Laplace transform is proportional to the reciprocal of the right-hand side of equation (D.11). Then

$$V(u; b) = h(u)/h'(b), \qquad 0 \le u \le b. \tag{D.15}$$

Let b^* denote the optimal value of b. Because of (D.15), it is the value of b that minimizes $h'(b)$. If $b^* > 0$, it satisfies the condition $h''(b^*) = 0$. For a general discussion on this problem, see Gerber, Lin, and Yang (2006).

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