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SOME OBSERVATIONS ON ACTUARIAL APPROXIMATIONS

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INTRODUCTION

APPROXIMATIONS in actuarial formulas are used because of the mathematically complex form of the mortality function and the fact that the number living is often defined only for integral values of age. The approximations used for different functions are selected for convenience and are often not consistent with each other. Different approximations imply different forms of the function l_{x+t} between integral values of age.

The method in this paper is to derive expressions for ${}_tq_x$ underlying the approximations employed and to compare such expressions. For two particular assumptions, namely, the assumption of linearity of l_{x+t}, hereinafter referred to as Basis A, and the assumption of linearity in the commutation function D_{x+t}, hereinafter referred to as Basis B, a comparison of the values of several annuity and insurance functions is made. A different approach is used in the last sections in examining the linearity of reserves between integral values of age.

The results are illustrated throughout with figures based on the 1958 CSO table with 3% interest. In this paper the variable t will be limited to the range $0 < t < 1$. Superfixes A, B, etc., are used to identify the bases being dealt with.

COMPARISON OF ${}_tq_x^A$ AND ${}_tq_x^B$

Basis A, which is usually referred to as assuming a uniform distribution of deaths within each year of age, is in general use for insurances payable at the moment of death. Basis B underlies the formulas in general use for annuities payable more frequently than annually.

By definition,

$$l^A_{x+t} = (1-t)l_x + tl_{x+1}$$

$$\therefore \quad {}_tp_x^A = 1 - tq_x$$

$$\therefore \quad {}_tq_x^A = tq_x .$$

By definition,

$$D^{B}_{x+t} = (1 - t)D_x + tD_{x+1}$$

$$\therefore \quad l^{B}_{x+t} = (1 + i)^t[(1 - t)l_x + tvl_{x+1}]$$

$$\therefore \quad {}_tp^{B}_x = (1 + i)^t[(1 - dt) - tvq_x]$$

$$\therefore \quad {}_tq^{B}_x = (1 + i)^t[tvq_x - (1 - dt) + v^t]$$

$$\therefore \quad {}_tq^{A}_x - {}_tq^{B}_x = (1 + i)^t[(1 - dt) - v^t] + tq_x[1 - v(1 + i)^t]$$

$$> 0 .$$

In conclusion, the l-curve traced by Basis B between two consecutive ages will lie above the l-curve traced by Basis A. Basis B will therefore produce higher annuity and lower insurance values than Basis A at integral ages.

RELATIVE ACCURACIES OF BASIS A AND BASIS B

In order to compare the relative accuracies of Basis A and Basis B it will be instructive to compare each with a third and presumably more accurate basis. The third basis which we shall select, hereinafter referred to as Basis S, is one which assumes that l^{S}_{x+t} is a third degree curve passing through l_x and l_{x+1} with slopes of $-l_x\mu_x$ and $-l_{x+1}\mu_{x+1}$ respectively.

It can be shown by testing the function and its derivative for $t = 0$ and $t = 1$ that

$$l^{S}_{x+t} = (2t^3 - 3t^2 + 1)l_x - (2t^3 - 3t^2)l_{x+1} - (t^3 - 2t^2 + t)(l_x\mu_x) - (t^3 - t^2)(l_{x+1}\mu_{x+1})$$

$$\therefore \quad {}_tp^{S}_x = 1 - tq_x + t(1 - t)[t\epsilon_2 - (1 - t)\epsilon_1]$$

$$\therefore \quad {}_tq^{S}_x = {}_tq^{A}_x - t(1 - t)[t\epsilon_2 - (1 - t)\epsilon_1] ,$$

where $\epsilon_1 = \mu_x - q_x$ and $\epsilon_2 = p_x\mu_{x+1} - q_x$.

To compare the values of ${}_tq_x$ on Bases A, B, and S we shall obtain expressions for the mean value $M({}_tq_x)$ of ${}_tq_x$ on each basis.

$$M({}_tq^{A}_x) = \int_0^1 tq_x dt$$

$$= \tfrac{1}{2} q_x .$$

$$M({}_tq^{B}_x) = \int_0^1 \{ [1 - (1+i)^t(1 - dt)] + (1+i)^t \cdot tvq_x \} dt$$

$$= \frac{\delta - d}{\delta^2} q_x - \frac{id - \delta^2}{\delta^2} .$$

$$M\left({}_tq_x^S\right) = \tfrac{1}{2}q_x - \int_0^1 t^2(1-t)\epsilon_2 - t(1-t)^2\epsilon_1\,dt$$

$$= \tfrac{1}{2}q_x - \tfrac{1}{12}(\epsilon_2 - \epsilon_1)\,.$$

In Table 1 the mean values of ${}_tq_x$ on the three bases are tabulated for several ages. Values of μ_x were obtained using the formula:

$$\mu_x \doteq \frac{7(d_{x-1}+d_x)-(d_{x-2}+d_{x+1})}{12l_x}\,.$$

The figures in Table 1 indicate that Basis A is in general more accurate than Basis B. We shall now compare the values of several actuarial functions on Bases A and B. Only the results are shown in the text, the development appearing in the appendix.

TABLE 1

Age	$1{,}000\mu_x$	$1{,}000\,p_x\mu_{x+1}$	$1{,}000\,\epsilon_1$	$1{,}000\,\epsilon_2$	$1{,}000\,M({}_tq_x^A)$	$1{,}000\,M({}_tq_x^B)$	$1{,}000\,M({}_tq_x^S)$
15.......	1.425	1.498	−.035	+.036	.730	.650	.724
30.......	2.106	2.157	−.024	+.027	1.065	.982	1.061
45.......	5.141	5.567	−.209	+.217	2.675	2.576	2.639
60.......	19.631	21.055	−.709	+.715	10.170	9.998	10.051
75.......	73.287	73.413	−.083	+.043	36.685	36.253	36.674

ANNUITIES PAYABLE MORE FREQUENTLY THAN ANNUALLY

For Basis A we have

$$a_x^{(m)A} = \frac{i-d}{i^{(m)}\cdot d^{(m)}}\cdot \ddot{a}_x - \frac{i-d^{(m)}}{i^{(m)}\cdot d^{(m)}}$$

$$\ddot{a}_x^{(m)A} = \frac{i-d}{i^{(m)}\cdot d^{(m)}}\cdot \ddot{a}_x - \frac{i-i^{(m)}}{i^{(m)}\cdot d^{(m)}}$$

$$\bar{a}_x^{A} = \frac{i-d}{\delta^2}\cdot \ddot{a}_x - \frac{i-\delta}{\delta^2}\,.$$

For Basis B we have

$$a_x^{(m)B} = \ddot{a}_x - \frac{m+1}{2m}$$

$$\ddot{a}_x^{(m)B} = \ddot{a}_x - \frac{m-1}{2m}$$

$$\bar{a}_x^{B} = \ddot{a}_x - \tfrac{1}{2}\,.$$

The approximations given by Basis B are the ones in general use. See formulas (2.18), (2.21), and (2.26) in Jordan's *Life Contingencies*.

By expanding the expression for $a_x^{(m)B} - a_x^{(m)A}$ as a power series in δ and ignoring terms higher than the second degree the following approximation to the difference is obtained:

$$a_x^{(m)B} - a_x^{(m)A} \doteq \frac{m^2-1}{6m^2}\cdot \delta\left[1+\frac{\delta}{4}-\frac{\delta}{2}\ddot{a}_x\right].$$

Similarly

$$\bar{a}_x^{B} - \bar{a}_x^{A} \doteq \frac{\delta}{6}\left[1+\frac{\delta}{4}-\frac{\delta}{2}\ddot{a}_x\right].$$

To obtain a measure of the errors in the annuity formulas the use of Basis S as a standard is not convenient. However, more accurate formulas than derived by Basis B can be obtained using Woolhouse's formula. By

TABLE 2—$\bar{a}_x$

Age	1,000 $\bar{a}_x^A$	Error in 1,000 $\bar{a}_x^A$	1,000 $\bar{a}_x^B$	Error in 1,000 $\bar{a}_x^B$
15.....	26,321.52	− .43	26,324.53	+2.58
30.....	22,974.37	− .61	22,977.62	+2.64
45.....	18,074.30	− .72	18,077.91	+2.89
60.....	12,130.67	+ .06	12,134.71	+4.10
75.....	6,643.97	+4.13	6,648.41	+8.57

using Woolhouse's formula a measure of the error in $a_x^{(m)B}$ is given by $[(m^2-1)/(12m^2)](\mu_x+\delta)$ and in $\bar{a}_x^B$ the error is $\frac{1}{12}(\mu_x+\delta)$. See Jordan's formula (2.17).

In Table 2 the error in $\bar{a}_x^B$ is compared with the error in $\bar{a}_x^A$ for several ages. The error in $a_x^{(m)}$ on the two bases is in the same proportion.

CONTINUOUSLY INCREASING ANNUITIES

For Basis A we have

$$(\bar{I}\bar{a})_x^A = \frac{i-d}{\delta^2}(Ia)_x - \frac{d(2+\delta)-i(2-\delta)}{\delta^3}\ddot{a}_x + \frac{i\delta-2(i-\delta)}{\delta^3}.$$

Expanding as a power series in δ, the following approximation is obtained:

$$(\bar{I}\bar{a})_x^A \doteq \left[1+\frac{\delta^2}{12}\right](Ia)_x - \left[\frac{\delta}{6}+\frac{\delta^2}{90}\right]\ddot{a}_x + \left[\frac{1}{6}+\frac{\delta}{12}+\frac{\delta^2}{40}\right].$$

For Basis B we have

$$(\bar{I}\bar{a})_x^B = (Ia)_x + \tfrac{1}{6}.$$

By using Woolhouse's formula the following accurate formula for the continuously increasing annuity is obtained

$$(\bar{I}\bar{a})_x = (Ia)_x + \tfrac{1}{12}\,.$$

See Jordan's formula (2.47). It is interesting to note how simple an accurate formula is in this instance. There is no incentive to use either Basis A or Basis B.

In Table 3 the error in $(\bar{I}\bar{a})^B$ is compared with the error in $(\bar{I}\bar{a})_x^A$ for several ages.

INSURANCES PAYABLE AT THE MOMENT OF DEATH

For Basis A we have

$$\bar{A}_x^A = \frac{i}{\delta} A_x$$

$$= 1 - \delta \bar{a}_x^A\,.$$

The approximation given by Basis A is one in general use. See Jordan's formula (3.18).

For Basis B we have

$$\bar{A}_x^B = 1 + \frac{\delta}{2} - \delta \ddot{a}_x$$

$$= 1 - \delta \bar{a}_x^B\,.$$

The error in $\bar{A}_x$ is $-\delta$ times the error in $\bar{a}_x$. Table 4 makes a comparison of $\bar{A}_x^A$ and $\bar{A}_x^B$.

TABLE 3—$(\bar{I}\bar{a})_x$

Age	1,000 $(\bar{I}\bar{a})_x^A$	Error	1,000 $(\bar{I}\bar{a})_x^B$	Error
15.....	543,097.67	− 6.80	543,187.81	+83.34
30.....	392,430.98	− 1.28	392,515.60	+83.34
45.....	235,529.18	+11.43	235,601.09	+83.34
60.....	107,625.08	+31.40	107,677.02	+83.34
75.....	34,436.91	+53.10	34,467.15	+83.34

TABLE 4—$\bar{A}_x$

Age	1,000 $\bar{A}_x^A$	Error	1,000 $\bar{A}_x^B$	Error
15.......	221.97	+.01	221.88	−.08
30.......	320.91	+.02	320.81	−.08
45.......	465.75	+.02	465.64	−.09
60.......	641.43	−.00	641.31	−.12
75.......	803.61	−.12	803.48	−.25

CONTINUOUSLY INCREASING INSURANCES

For Basis A we have

$$(\bar{I}\bar{A})^{A}_{x} = \frac{i}{\delta}\left[(IA)_{x} - \frac{\delta - d}{\delta d} A_{x}\right]$$

$$= \bar{a}^{A}_{x} - \delta\,(\bar{I}\bar{a})^{A}_{x} .$$

For Basis B we have

$$(\bar{I}\bar{A})^{B}_{x} = \ddot{a}_{x} - \delta\,(Ia)_{x} - \left(\frac{1}{2} + \frac{\delta}{6}\right)$$

$$= \bar{a}^{B}_{x} - \delta\,(\bar{I}\bar{a})^{B}_{x} .$$

A formula commonly used in practice, which we shall call $(\bar{I}\bar{A})^{P}_{x}$, is given by Jordan's formula (3.30):

$$(\bar{I}\bar{A})^{P}_{x} = \frac{\bar{R}^{A}_{x} - \frac{1}{2}\bar{M}^{A}_{x}}{D_{x}}$$

$$= \frac{i}{\delta}\left[\frac{R_{x} - \frac{1}{2}M_{x}}{D_{x}}\right]$$

$$= \frac{i}{\delta}\left[(IA)_{x} - \tfrac{1}{2}A_{x}\right]$$

$$= (\bar{I}\bar{A})^{A}_{x} + \frac{i}{\delta}\left[\frac{\delta - d}{\delta d} - \frac{1}{2}\right]A_{x} \doteq (\bar{I}\bar{A})^{A}_{x} + \frac{i}{12} A_{x} .$$

The errors in $(\bar{I}\bar{A})^{A}_{x}$ and $(\bar{I}\bar{A})^{B}_{x}$ are equal to the corresponding error in $\bar{a}_{x}$ less δ times the corresponding error in $(\bar{I}\bar{a})_{x}$. The error in $(\bar{I}\bar{A})^{P}_{x}$ is obtained by adding $(i/12)A_{x}$ to the error in $(\bar{I}\bar{A})^{A}$. Table 5 makes a comparison of $(\bar{I}\bar{A})^{A}_{x}$, $(\bar{I}\bar{A})^{B}_{x}$ and $(\bar{I}\bar{A})^{P}_{x}$.

TABLE 5

Age	1,000 $(\bar{I}\bar{A})^{A}_{x}$	Error	1,000 $(\bar{I}\bar{A})^{B}_{x}$	Error	1,000 $(\bar{I}\bar{A})^{P}_{x}$	Error
15.......	10,268.20	− .23	10,268.55	+ .12	10,268.75	+ .32
30.......	11,374.58	− .57	11,375.33	+ .18	11,375.37	+ .22
45.......	11,112.34	−1.06	11,113.82	+ .43	11,113.49	+ .09
60.......	8,949.40	− .87	8,951.91	+1.64	8,950.98	+ .71
75.......	5,626.06	+2.56	5,629.60	+6.11	5,628.04	+4.54

ASSUMPTION OF LINEARITY OF $1/l_{x+t}$ REFERRED TO AS BASIS C

By definition

$$\frac{1}{l^{C}_{x+t}} = \frac{1-t}{l_{x}} + \frac{t}{l_{x+1}}$$

$$\therefore l^{C}_{x+t} = \frac{l_x l_{x+1}}{t l_x + (1-t)\, l_{x+1}}$$

$$\therefore {}_t q^{C}_x = \frac{t q_x}{1-(1-t)\, q_x}$$

$$\doteq t q_x + t(1-t)\, q_x^2$$

$$\therefore M({}_t q^{C}_x) \doteq \tfrac{1}{2} q_x + \tfrac{1}{6} q_x^2 .$$

Table 6 below compares $M({}_tq^C_x)$ and $M({}_tq^A_x)$. It will be seen that the l-curve traced by Basis C lies below the one traced by Basis A.

It can be shown that ${}_{1-t}q^C_{x+t} = (1-t)q_x$.

This is the well-known Balducci hypothesis used in exposure work.

TABLE 6

Age	1,000 $M({}_tq^A_x)$	1,000 $M({}_tq^C_x)$
15	.730	.730
30	1.065	1.066
45	2.675	2.680
60	10.170	10.239
75	36.685	37.582

ASSUMPTION OF LINEARITY OF $1/D_{x+t}$ REFERRED TO AS BASIS D

By definition

$$\frac{1}{D^{D}_{x+t}} = \frac{1-t}{D_x} + \frac{t}{D_{x+1}}$$

$$\therefore {}_{1-t}q^{D}_{x+t} = v^t (1-t)\, q_x - [v^t(1+ti) - 1]$$

$$< {}_{1-t}q^{C}_{x+t}$$

$$\therefore {}_{1-t}p^{D}_{x+t} > {}_{1-t}p^{C}_{x+t}$$

$$\therefore {}_tp^{D}_x < {}_tp^{C}_x .$$

In conclusion the l-curve traced by Basis D lies below that traced by Basis C, and hence below those traced by Bases A and B.

Basis D is used in the approximation

$$_{1-t|}\ddot{a}^{D}_{x+t} = (1-t)\, a_x + t\ddot{a}_{x+1} .$$

As Basis D understates the value of l_{x+t} it will overstate the value of $1/D_{x+t}$. The above approximation therefore overstates results.

ASSUMPTION OF LINEARITY OF RESERVES

The assumption of linearity in reserves between integral values of duration is general and accounts for the usual formulas for mean reserves. As we shall be considering the reserve during one policy year only, we shall, for convenience, use the symbol V_t to represent the reserve at time t during the year for which the initial reserve is V_0 and the terminal reserve is V_1. Premiums are assumed to be payable annually.

Our approach will be to examine the first and second derivatives of the reserve formula and then to make some observations about the curve traced by the reserve during the year. We can then draw some conclusions about the assumption of linearity in reserves. We shall first assume the payment of claims at the moment of death.

Reserve during Policy Year Assuming Claims Payable at Moment of Death

The reserve can be evaluated from the following retrospective equation.

$$V_t = \frac{V_0 \cdot D_x - (\bar{M}_x - \bar{M}_{x+t})}{D_{x+t}}.$$

The derivatives are as follows:

$$\frac{dV_t}{dt} = V_t(\mu + \delta) - \mu$$

$$\frac{d^2V_t}{dt^2} = V_t[(\mu + \delta)^2 + \mu'] - [\mu(\mu + \delta) + \mu'],$$

where μ is the force of mortality and μ' is the derivative of the force of mortality at time t.

The first derivative is positive or negative and the reserve will increase or decrease according as

$$V_t \gtrless \frac{\mu}{\mu + \delta} = K_1.$$

K_1 is the critical value of the reserve where the interest earned is just sufficient to pay claims without encroaching on the reserve.

The second derivative is positive or negative and the reserve slope will increase or decrease according as

$$V_t \gtrless \frac{\mu(\mu + \delta) + \mu'}{(\mu + \delta)^2 + \mu'} = K_2.$$

It can be shown that

$$K_2 = \frac{K_1 + a}{1 + a} = K_1 + \frac{a}{1 + a}(1 - K_1)$$

where $a = \mu'/(\mu + \delta)^2$. It will be evident that $K_2 > K_1$ as long as the force of mortality is increasing.

At any point the reserve and its slope will both be increasing if the reserve exceeds K_2. The reserve will be increasing but with decreasing slope if it lies between K_1 and K_2. Both the reserve and its slope will be decreasing if the reserve is below K_1.

For a whole policy year, however, the analysis is complicated by the fact that K_1 and K_2 are variables. The conclusions made about the behavior of the reserve at the beginning of the year might not be valid throughout the year.

As long as the force of mortality is increasing, K_1 will be an increasing function and although K_2 may not necessarily increase when K_1 is increasing it must in general be an increasing function in order to exceed K_1. The usual situation then is one in which both K_1 and K_2 increase through the year.

If the reserve at the beginning of the year is much larger than K_2, the reserve will increase with increasing slope throughout the whole year since it will be greater than K_2 throughout the whole year. In such cases the reserve at midyear will be less than the mean of initial and terminal reserves and the usual mean reserve formula will overstate results.

If the reserve at the beginning of the year is less than K_1, both reserve and slope will decrease throughout the whole year, resulting in a plunging effect. For such cases the reserve at midyear will be greater than the mean of initial and terminal reserves and the usual mean reserve formula will understate results.

If the reserve at the beginning of the year lies between K_1 and K_2, it will start to increase with decreasing slope. Unless further premiums are paid the reserve will reach a maximum and start to plunge, although the maximum will not necessarily occur in the current year. For such cases the reserve at midyear will be greater than the mean of initial and terminal reserve and the usual mean reserve formula will understate results.

Finally, if the reserve at the beginning of the year exceeds K_2, it will start to increase with increasing slope. If the initial reserve is not sufficiently greater than K_2, the rate of increase will be slow and at a later point the reserve may fail to be greater than K_2. When such a stage is reached the reserve will commence a stage of increasing with decreasing slope and eventually will plunge unless further premiums are paid. For such cases it is not possible, without a table of values of K_1 and K_2, to draw conclusions about the accuracy of the usual mean reserve.

The analysis of the reserve during the year would be improved if values of K_1, K_2, and the reserve at the beginning and end of the year are available.

Table 7 gives the values of K_1 and K_2 on the 1958 CSO table with 3% interest for a number of ages. The values of μ'_x were developed from the following approximate formula accurate to fourth differences:

$$\mu'_x \doteq \frac{15\,(d_x - d_{x-1}) - (d_{x+1} - d_{x-2})}{12 l_x}\,.$$

Reserve during Policy Year Assuming Claims Payable at the End of the Year of Death

Although the assumption that claims are payable at the end of the policy year is artificial, it is commonly used in the calculation of reserves for practical reasons and by custom. Such an assumption is equivalent to assuming an increasing death benefit during each year, since the payment of benefits on claims occurring early in a year is postponed for the longest period. Although such an assumption has a relatively small effect on the values of the reserves themselves and on the value of K_1, the effect on the value of K_2 is remarkable.

TABLE 7

Age	1,000 μ_x	1,000 K_1	1,000 μ'_x	1,000 K_2
15......	1.425	45.99	.067	108.41
30......	2.106	66.51	.045	106.48
45......	5.141	148.16	.405	362.55
60......	19.631	399.09	1.400	619.36
75......	73.287	712.59	.263	719.56

The reserve can be evaluated from the following prospective equation:

$$V_t = v^{1-t}\,[V_1 + {}_{1-t}q_{x+t}\,(1 - V_1)\,]$$

$$\frac{dV_t}{dt} = V_t\,(\mu + \delta) - v^{1-t}\mu$$

$$\frac{d^2V_t}{dt^2} = V_t\,[\,(\mu + \delta)^2 + \mu'] - v^{1-t}\,[\mu\,(\mu + 2\,\delta) + \mu']$$

$$\therefore K_1 = v^{1-t}\,\frac{\mu}{\mu + \delta}$$

$$K_2 = \frac{v^{1-t}\,[\mu\,(\mu + 2\,\delta) + \mu']}{(\mu + \delta)^2 + \mu'}\,.$$

In Table 8 we show values of K_1 and K_2 for a number of ages. It is interesting to compare these values with Table 7.

CONCLUSION

In the introduction it was stated that actuarial approximations are often inconsistent with each other. Such inconsistencies are tolerated because in practical work extreme accuracy is unnecessary. However, there is one assumption that is artificial and dispensable, namely, the assumption of the payment of claims at the end of the year of death. The publishing of commutation columns assuming payment of claims at the moment of death would be consistent with common practice and there would appear to be no real need for the commutation functions C_x, M_x, R_x.

TABLE 8

Age	1,000 K_1	1,000 K_2
15	44.65	145.07
30	64.57	161.08
45	143.84	443.68
60	387.47	748.81
75	691.83	892.62

APPENDIX

Annuities Payable More Frequently Than Annually

$$a_x^{(m)} = \sum_{h=1}^{\infty} v^{h/m} \cdot {}_{h/m}p_x$$

$$= \frac{D_x^{(m)} + D_{x+1}^{(m)} + \ldots}{D_x}$$

$$= \frac{N_x^{(m)}}{D_x},$$

where

$$D_x^{(m)} = \frac{1}{m} \sum_{h=1}^{m} D_{x+h/m}$$

and

$$N_x^{(m)} = \sum_{s=0}^{\infty} D_{x+s}^{(m)}.$$

For Basis A

$$D_{x+t}^{A} = v^{x+t} l_{x+t}^{A}$$

$$= v^t [(1-t) D_x + t (1+i) D_{x+1}]$$

$$D_x^{(m)A} = \frac{1}{m}\sum_{h=1}^{m} D^A_{x+h/m}$$

$$= \frac{1}{m}\sum_{h=1}^{m} v^{h/m}\left[\left(1-\frac{h}{m}\right) D_x + \frac{h}{m}(1+i)\, D_{x+1}\right]$$

$$= \frac{d^{(m)}-d}{i^{(m)}\cdot d^{(m)}}\, D_x + \frac{i-d^{(m)}}{i^{(m)}\cdot d^{(m)}}\, D_{x+1}$$

$$\therefore N_x^{(m)A} = \frac{i-d}{i^{(m)}\cdot d^{(m)}}\, N_x - \frac{i-d^{(m)}}{i^{(m)}\cdot d^{(m)}}\, D_x$$

$$\therefore a_x^{(m)A} = \frac{i-d}{i^{(m)}\cdot d^{(m)}}\, \ddot{a}_x - \frac{i-d^{(m)}}{i^{(m)}\cdot d^{(m)}}.$$

For Basis B

$$D_x^{(m)B} = \frac{1}{m}\sum_{h=1}^{m} D^B_{x+h/m}$$

$$= \frac{1}{m}\sum_{h=1}^{m}\left[\left(1-\frac{h}{m}\right)D_x + \frac{h}{m}\, D_{x+1}\right]$$

$$= \frac{m-1}{2m}\, D_x + \frac{m+1}{2m}\, D_{x+1}$$

$$\therefore N_x^{(m)B} = N_x - \frac{m+1}{2m}\, D_x$$

$$\therefore a_x^{(m)B} = \ddot{a}_x - \frac{m+1}{2m}.$$

Continuously Increasing Annuities

$$(\bar{I}\bar{a})_x = \int_0^\infty s\, v^s\, {}_sp_x\, ds$$

$$= \frac{\bar{S}_x - \bar{G}_x}{D_x},$$

where

$$\bar{S}_x = \sum_{n=0}^{\infty} \bar{N}_{x+}$$

$$\bar{G}_x = \sum_{n=0}^{\infty} \bar{F}_{x+n}$$

$$\bar{F}_x = \int_0^1 (1-t)\, D_{x+t}\,dt\,.$$

For Basis A

$$\bar{N}_x^A = \frac{i-d}{\delta^2} N_x - \frac{i-\delta}{\delta^2} D_x$$

$$\therefore \bar{S}_x^A = \frac{i-d}{\delta^2} S_x - \frac{i-\delta}{\delta^2} N_x$$

$$= \frac{i-d}{\delta^2} S_{x+1} + \frac{\delta-d}{\delta^2} N_x$$

$$\bar{F}_x^A = \int_0^1 (1-t)\, D_{x+t}^A\,dt$$

$$= \int_0^1 v^t (1-t) [\,(1-t)\, D_x + t\,(1+i)\, D_{x+1}]\,dt$$

$$= \frac{\delta^2 - 2\,(\delta-d)}{\delta^3} D_x + \frac{i\,\delta - 2\,(i-\delta)}{\delta^3} D_{x+1}$$

$$\therefore \bar{G}_x^A = \frac{\delta\,(i+\delta) - 2\,(i-d)}{\delta^3} N_x - \frac{i\,\delta - 2\,(i-\delta)}{\delta^3} D_x$$

$$\therefore \bar{S}_x^A - \bar{G}_x^A = \frac{i-d}{\delta^2} S_{x+1} - \frac{d\,(2+\delta) - i\,(2-\delta)}{\delta^3} N_x + \frac{i\,\delta - 2(i-\delta)}{\delta^3} D_x$$

$$\therefore (\bar{I}\bar{a})_x^A = \frac{i-d}{\delta^2} (Ia)_x - \frac{d\,(2+\delta) - i\,(2-\delta)}{\delta^3} \bar{a}_x + \frac{i\,\delta - 2\,(i-\delta)}{\delta^3}\,.$$

For Basis B

$$\bar{N}_x^B = N_x - \tfrac{1}{2} D_x$$

$$\therefore \bar{S}_x^B = S_x - \tfrac{1}{2} N_x$$

$$= S_{x+1} + \tfrac{1}{2} N_x$$

$$\bar{F}_x^B = \int_0^1 (1-t)\, D_{x+t}^B\,dt$$

$$= \int_0^1 (1-t) [\,(1-t)\, D_x + t D_{x+1}]\, d$$

$$= \tfrac{1}{3} D_x + \tfrac{1}{6} D_{x+1}$$

$$\therefore \bar{G}^B_x = \tfrac{1}{2} N_x - \tfrac{1}{6} D_x$$

$$\therefore \bar{S}^B_x - \bar{G}^B_x = S_{x+1} + \tfrac{1}{6} D_x$$

$$\therefore (\bar{I}\bar{a})^B_x = (Ia)_x + \tfrac{1}{6}\,.$$

Insurances Payable at the Moment of Death

$$\bar{A}_x = \int_0^\infty v^s {}_s p_x \mu_{x+s} d\,s$$

$$= \frac{\bar{M}_x}{D_x},$$

where

$$\bar{M}_x = \sum_{n=0}^{\infty} \bar{C}_{x+n}$$

and

$$\bar{C}_x = \int_0^1 D_{x+t}\mu_{x+t} dt\,.$$

For Basis A

$$\bar{C}^A_x = \int_0^1 D^A_{x+t}\mu^A_{x+t} dt$$

$$= \int_0^1 v^t (1+i) C_x dt$$

$$= \frac{i}{\delta} C_x$$

$$\therefore \bar{M}^A_x = \frac{i}{\delta} M_x$$

$$\therefore \bar{A}^A_x = \frac{i}{\delta} A_x$$

$$= 1 - \delta \bar{a}^A_x\,.$$

For Basis B

$$\bar{C}^B_x = \int_0^1 D^B_{x+t}\mu^B_{x+t} dt$$

$$= \int_0^1 [1 - \delta(1-t)]\, D_x - [1 + \delta t]\, D_{x+1} dt$$

$$= \left(1 - \frac{\delta}{2}\right) D_x - \left(1 + \frac{\delta}{2}\right) D_{x+1}$$

$$\therefore \bar{M}_x^B = \left(1 + \frac{\delta}{2}\right) D_x - \delta N_x$$

$$\therefore \bar{A}_x^B = 1 + \frac{\delta}{2} - \delta \ddot{a}_x$$

$$= 1 - \delta \bar{a}_x^B .$$

Continuously Increasing Insurances

$$(\bar{I}\bar{A})_x = \int_0^\infty t\, v^t\, {}_tp_x \mu_{x+t} dt$$

$$= \frac{\bar{R}_x - \bar{K}_x}{D_x} ,$$

where

$$\bar{R}_x = \sum_{n=0}^{\infty} \bar{M}_{+n}$$

and

$$\bar{K}_x = \sum_{n=0}^{\infty} \bar{J}_{x+n}$$

$$\bar{J}_x = \int_0^1 (1 - t)\, D_{x+t} \mu_{x+t} dt .$$

For Basis A

$$\bar{R}_x^A = \sum_{n=0}^{\infty} \bar{M}_{x+n}^A$$

$$= \frac{i}{\delta} \cdot R_x$$

$$\bar{J}_x^A = \int_0^1 (1 - t)\, D_{x+t}^A \mu_{x+t}^A dt$$

$$= \int_0^1 (1 - t)\, v^t (1 + i)\, C_x d$$

$$= \frac{\delta (1 + i) - i}{\delta^2} C_x$$

$$\therefore \bar{K}_x^A = \frac{\delta(1+i) - i}{\delta^2} M_x$$

$$\therefore \bar{R}_x^A - \bar{K}_x^A = \frac{i}{\delta}\left[R_x - \frac{\delta - d}{\delta d} M_x\right]$$

$$\therefore (\bar{I}\bar{A})_x^A = \frac{i}{\delta}\left[(IA)_x - \frac{\delta - d}{\delta d} A_x\right].$$

For Basis B

$$\bar{R}_x^B = \sum_{n=0}^{\infty} \bar{M}_{x+n}^B$$

$$= \left(1 + \frac{\delta}{2}\right) N_x - \delta S_x$$

$$\bar{J}_x^B = \int_0^1 (1-t)\, D_{x+t}^B \mu_{x+t}^B\, dt$$

$$= \int_0^1 \{(1-t)[1 - \delta(1-t)] D_x - (1-t)(1+\delta t) D_{x+1}\}\, dt$$

$$= \left[\frac{1}{2} - \frac{\delta}{3}\right] D_x - \left[\frac{1}{2} + \frac{\delta}{6}\right] D_{x+1}$$

$$\therefore \bar{K}_x^B = D_x\left[\frac{1}{2} + \frac{\delta}{6}\right] - \frac{\delta}{2} N_x$$

$$\therefore \bar{R}_x^B - \bar{K}_x^B = (1+\delta) N_x - \delta S_x - \left(\frac{1}{2} + \frac{\delta}{6}\right)$$

$$\therefore (\bar{I}\bar{A})_x^B = (1+\delta)\ddot{a}_x - \delta(I\ddot{a})_x - \left(\frac{1}{2} + \frac{\delta}{6}\right)$$

$$= \ddot{a}_x - \delta(Ia)_x - \left(\frac{1}{2} + \frac{\delta}{6}\right).$$

DISCUSSION OF PRECEDING PAPER

HARRY M. SARASON:

Mr. Mercu has discussed approximations from a strictly actuarial viewpoint. In certain important areas, however, actuarial computations have a precise legal meaning. When an actuary makes computations which have an exact legal meaning, he is not making an approximate calculation: he is making an exact calculation—actuarially approximate, but legally exact.

Laws, regulations and judicial decisions establish straight line interpolation as an exact method of interpolating cash values, single premiums for paid-up equivalents, and mean reserves. Legally the present values of each day of extended insurance in a year of age are identical. Actuarially, these calculation methods may be looked upon as approximations, but the laws have the last and strongest word—actuarial refinements involving μ_x and δ are not, legally, refined approximations. Legally either they give legal values and are legally exact or else they don't give legal values and are wrong.

C. J. NESBITT:

On reading this paper, I was reminded of an approximation basis which Mrs. Butcher and I encountered in our paper, "Rate Functions and their Role in Actuarial Mathematics," *RAIA* XXXVII, 202 (see formulas [46] and [47]). This basis, which I shall refer to as Basis *, was obtained by considering D_{x+t} as a function of two decrements (discount and mortality) and for $0 < t < 1$ assuming uniformity in respect to each decrement, or equivalently, linearity of v^{x+t} and l_{x+t}. Another way of expressing it is to say that in addition to the assumption of uniform distribution of deaths one assumes simple discount, in each year of age. Thus:

$$D^*_{x+t} = v^x(1-td)l_x(1-tq_x) = D_x(1-td)(1-tq_x).$$

Comparing with D^A_{x+t} of the paper, we have

$$D^*_{x+t}/D^A_{x+t} = (1+i)^t(1-td) = (1-td)/(1-d)^t;$$

and since, for $0 < t < 1$ and a given rate of discount, simple discount present values exceed those under compound discount, it follows that

$$D^*_{x+t} > D^A_{x+t}, \qquad 0 < t < 1.$$

Basis * will then give higher annuity and lower insurance values than Basis A at integral ages.

To show that Basis * is not entirely unfamiliar, I define

$$C_x^{(m)} = \sum_{h=0}^{m-1} v^{x+(h+1)/m}\left(l_{x+h/m} - l_{x+(h+1)/m}\right).$$

Then

$$C_x^{(m)*} = \sum_{h=0}^{m-1} v^x\left(1 - \frac{h+1}{m}\cdot d\right)\frac{d_x}{m}$$

$$= v^x d_x\left(1 - \frac{m+1}{2m}\cdot d\right) = v^{x+1} d_x\left(1 + i - \frac{m+1}{2m}\cdot i\right)$$

$$= \left(1 + \frac{m-1}{2m}\cdot i\right)C_x .$$

It follows that

$$M_x^{(m)*} = \left(1 + \frac{m-1}{2m}\cdot i\right)M_x$$

$$\bar{C}_x^* = \lim_{m\to\infty} C_x^{(m)*} = \left(1 + \tfrac{1}{2}i\right)C_x$$

$$\bar{M}_x^* = \left(1 + \tfrac{1}{2}i\right)M_x$$

or

$$\bar{A}_x^* = \left(1 + \tfrac{1}{2}i\right)A_x ,$$

the last two of which formulas are frequently used in practical work.

Less familiar are the corresponding approximations for annuity values. These may be obtained as follows:

$$N_x^{(m)*} = \frac{1}{d^{(m)}}\left[D_x - M_x^{(m)*}\right]$$

$$= \frac{1}{d^{(m)}}\left[N_x - N_{x+1} - \left(1 + \frac{m-1}{2m}\cdot i\right)\left(vN_x - N_{x+1}\right)\right]$$

$$= \frac{m+1}{2m}\cdot\frac{d}{d^{(m)}}\cdot N_x + \frac{m-1}{2m}\cdot\frac{i}{d^{(m)}}\cdot N_{x+1} .$$

$$\ddot{a}_x^{(m)} = \frac{m+1}{2m}\cdot\frac{d}{d^{(m)}}\cdot \ddot{a}_x + \frac{m-1}{2m}\cdot\frac{i}{d^{(m)}}\cdot a_x$$

and

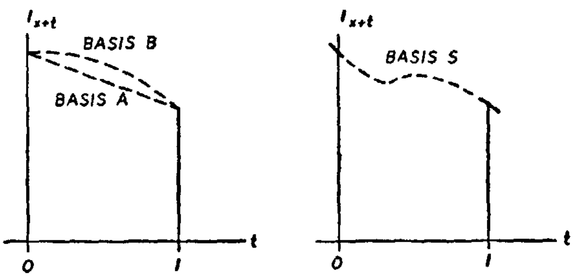
$$\bar{a}_x = \frac{1}{2\delta}\left(d\ddot{a}_x + i a_x\right).$$

As such, these formulas appear as modifications of the formulas given by the author's Basis B, but could also be considered in relation to his Basis A.

Final examinations preclude further exploration of this interesting paper. The author is to be congratulated on the ingenuity and thoroughness of his analysis. At the very least, the paper will provide a fruitful source for class discussion, and we thank him for it.

GEOFFREY CROFTS:

When students (and others) are confronted with more than one approximation for functions which arise from the same premise, the question invariably arises, "Which is best?" The answer is, "It depends on the truth." It could be possible that one of the approximations actually is the truth. However, the questioner is not usually satisfied with this



answer. What he wants is a comparison with the truth (which is usually impossible) or with a model which intuitively appeals to him as being much closer to the truth than the approximations. Mr. Mereu has been rather ingenious in constructing models which have this appeal.

I find that a graphic method goes a little further in demonstrating the nature of the approximations and the comparison model. The questioner can graphically supply the truth any way he sees fit. He is able to judge under what condition one approximation would be better than another.

Mr. Mereu's bases A, B, and S could be demonstrated by drawing the l_x curve for each basis as shown in the accompanying graphs.

By using a little imagination or with careful construction it is possible to consider various results for Basis S with different given slopes. I have shown a rather unusual case for Basis S in which the slope is the same at $t = 0$ and $t = 1$ but of such steepness that if the curve continued to decrease at this rate for the whole year it would be lower than the given height at $t = 1$. A more usual case would be one in which the slope at $t = 0$ is not as steep as the slope of the line joining the heights at $t = 0$

and $t = 1$; and the slope at $t = 1$ is steeper than the slope of such line. Mr. Mereu determines the slopes to be used in his comparison by intuitively appealing models taking account of the heights of the curve at other integral values of the argument.

His method of analyzing the linearity of reserve assumption is also clever. I have one comment here. He states that if the reserve at the beginning of the year is between K_1 and K_2, the reserve will reach a maximum and start to plunge unless further premiums are paid. Is it not possible to conceive of a function increasing indefinitely with a continually decreasing slope?

We now have another answer to the question, "Which approximation is best?" That answer is "See Mereu's paper."

MARJORIE V. BUTCHER:

To me, studying Mr. Mereu's paper has been a fascinating adventure in life contingencies, and I unqualifiedly recommend it to every student of the subject. The author skillfully explores and compares the effects on basic functions of some traditional actuarial approximations. These are linearity assumptions for $0 < t < 1$ of each of the following: l_{x+t}, D_{x+t}, $1/l_{x+t}$, $1/D_{x+t}$ and reserves V_t at fractional durations. My comments which follow are offered in a spirit of appreciation for what Mr. Mereu has accomplished.

One could add μ_{x+t} to his group of basic functions. The various bases of the paper produce

$$\mu^{A}_{x+t} = \frac{q_x}{1 - t\,q_x}, \tag{1}$$

which is Jordan's formula (1.24),

$$\mu^{B}_{x+t} = v \cdot \frac{i + q_x}{1 - dt - vt\,q_x} - \delta = v^{1-t} \cdot \frac{i + q_x}{{}_tp^{B}_x} - \delta, \tag{2}$$

$$\mu^{C}_{x+t} = \frac{q_x}{1 - (1 - t)\,q_x} = {}_tq^{C}_x / t \tag{3}$$

and

$$\mu^{D}_{x+t} = \frac{i + q_x}{1 + it - (1 - t)\,q_x} - \delta\,. \tag{4}$$

Although ${}_tq^{A}_x < {}_tq^{C}_x$ whenever $0 < t < 1$,

$$\mu^{A}_{x+t} / \mu^{C}_{x+t} \begin{cases} < 1\,, & 0 < t < \frac{1}{2} \\ = 1\,, & t = \frac{1}{2} \\ > 1\,, & \frac{1}{2} < t < 1\,. \end{cases}$$

The same relations hold for $\mu^{B}_{x+t} / \mu^{D}_{x+t}$.

It appears to me that in his Appendix Mr. Mereu gives an unusual definition to $N_x^{(m)}$, essentially

$$N_x^{(m)} = \frac{1}{m} \sum_{h=1}^{\infty} D_{x+h/m} .$$

This commutation symbol is not defined in the international code and is somewhat obscurely placed in Jordan, where it is defined as

$$N_x^{(m)} = N_x - \frac{m-1}{2m} \cdot D_x .$$

However, it seems preferable to have

$$N_x^{(m)} = \frac{1}{m} \sum_{h=0}^{\infty} D_{x+h/m} . \qquad (5)$$

An advantage of definition (5) is that it produces the standard

$$N_x = \sum_{h=0}^{\infty} D_{x+h}$$

when $m = 1$; and

$$\ddot{a}_x^{(m)} = N_x^{(m)} / D_x$$

is an exact formula, although it is generally impossible to calculate $N_x^{(m)}$ exactly. The only formulas affected in the paper are those in the Appendix for $a_x^{(m)}$, and $D_x^{(m)} = N_x^{(m)} - N_{x+1}^{(m)}$ and $N_x^{(m)}$ on Bases A and B. The adjustments are, of course, a simple matter.

To be consistent, approximations for $\ddot{a}_x^{(m)}$ and $A_x^{(m)}$ must satisfy the equation

$$1 = d^{(m)} \ddot{a}_x^{(m)} + A_x^{(m)} . \qquad (6)$$

Equivalently,

$$D_x = d^{(m)} N_x^{(m)} + M_x^{(m)} , \qquad (7)$$

where $N_x^{(m)}$ is given by (5) and

$$M_x^{(m)} = \sum_{h=0}^{\infty} v^{x+(h+1)/m} (l_{x+h/m} - l_{x+(h+1)/m}) . \qquad (8)$$

There are analogues for the continuous case and the cases of increasing annuities and insurances. Once a convenient approximation, say for an a or N, is found, use of one of the preceding formulas yields the consistent A or M. Thus from

$$\ddot{a}_x^{(m)\,B} = \ddot{a}_x - \frac{m-1}{2m}$$

and equation (6),

$$A_x^{(m)B} = 1 - d^{(m)}\left(\ddot{a}_x - \frac{m-1}{2m}\right).$$

The same method furnishes a convenient way of determining $\ddot{a}_x^{(m)A}$, by first finding $A_x^{(m)A}$, where

$$A_x^{(m)} = \frac{1}{l_x}\sum_{h=0}^{\infty} v^{(h+1)/m}(l_{x+h/m} - l_{x+(h+1)/m}). \qquad (9)$$

Now the Basis A assumption of linearity of l_{x+t}, $0 < t < 1$, implies that

$$l_{x+h/m} - l_{x+(h+1)/m} = \frac{1}{m}\cdot d_x, \quad 0 \leq h < m, \qquad (10)$$

i.e., that deaths are uniformly distributed within each year of age. Accordingly,

$$C_y^{(m)} = M_y^{(m)} - M_{y+1}^{(m)} = \sum_{h=0}^{m-1} v^{y+(h+1)/m}(l_{y+h/m} - l_{y+(h+1)/m})$$

becomes

$$C_y^{(m)A} = (v^{y+1}d_y)\frac{1}{m}\sum_{h=0}^{m-1}(1-i)^{1-(h+1)/m}$$

$$= \frac{i}{i^{(m)}}C_y .$$

Then

$$A_x^{(m)A} = \frac{1}{D_x}\sum_{y=x}^{\infty} C_y^{(m)A} = \frac{i}{i^{(m)}}A_x . \qquad (11)$$

By use of (6),

$$\ddot{a}_x^{(m)A} = \frac{1}{d^{(m)}}\left(1 - \frac{i}{i^{(m)}}A_x\right), \qquad (12)$$

which upon substitution of $1 - d\ddot{a}_x$ for A_x readily yields the result in the paper.

Another familiar approximation arises from combining the assumption of linearity of the l-function within a year of age with the assumption of simple discount in the year of death (Basis E). Here

$$C_y^{(m)E} = \sum_{h=0}^{m-1} v^y\left(1 - \frac{h+1}{m}\cdot d\right)\frac{d_y}{m} \qquad (13)$$

$$= \left(1 + \frac{m-1}{2m}\cdot i\right)C_y ,$$

so that

$$A_x^{(m)E} = \left(1 + \frac{m-1}{2m}\cdot i\right) A_x \tag{14}$$

and

$$\bar{A}_x^E = \left(1 + \frac{i}{2}\right) A_x . \tag{15}$$

The corresponding consistent forms for annuities are

$$\ddot{a}_x^{(m)E} = \frac{1}{d^{(m)}}\left(\frac{m-1}{2m}\cdot i + \frac{m+1}{2m}\cdot d\right) \ddot{a}_x - \frac{m-1}{2m}\cdot\frac{i}{d^{(m)}} \tag{16}$$

and

$$\bar{a}_x^E = \frac{i+d}{2\delta}\cdot \ddot{a}_x - \frac{i}{2\delta} . \tag{17}$$

By extension of the preceding method $(I^{(m)} A)_x^{(m)E}$, $(\bar{I}\bar{A})_x^E$, $(I^{(m)} \ddot{a})_x^{(m)E}$ and $(\bar{I}\bar{a})_x^E$ are expressible.

Another common approximation (Basis F) is

$$\bar{A}_x^F = (1+i)^{1/2} A_x , \tag{18}$$

coupled with the consistent but unfamiliar

$$\bar{a}_x^F = \frac{1}{\delta}\left[1 - (1+i)^{1/2} A_x\right]$$

$$= v^{1/2}\cdot\frac{i}{\delta}\cdot \ddot{a}_x - \bar{s}_{\overline{1/2}|} . \tag{19}$$

The set of inequalities

$$\bar{A}_x^B < \bar{A}_x^F < \bar{A}_x^A < \bar{A}_x^E \tag{20}$$

result after expansion (in powers of δ) of the interest factors in the various approximations to $\bar{A}_x$. For $\bar{a}_x$ the inequalities are reversed. The accompanying tables extend Tables 4 and 2 of the paper.

TABLE I—$\bar{A}_x$

Age	1,000 $\bar{A}_x$	Error in Approximation to 1,000 $\bar{A}_x$			
		Basis B	Basis F	Basis A	Basis E
15	221.96	−.08	+.00	+.01	+.03
30	320.89	−.08	+.01	+.02	+.04
45	465.73	−.09	+.00	+.02	+.05
60	641.43	−.12	−.02	−.00	+.05
75	803.73	−.25	−.15	−.12	−.06

TABLE II—$\bar{a}_x$

Age	1,000 $\bar{a}_x$	Error in Approximation to 1,000 $\bar{a}_x$			
		Basis B	Basis F	Basis A	Basis E
15	26,321.95	+2.58	− .15	− .43	− .97
30	22,974.98	+2.64	− .22	− .61	−1.40
45	18,075.02	+2.89	− .15	− .72	−1.87
60	12,130.61	+4.10	+ .85	+ .06	−1.52
75	6,639.84	+8.57	+5.12	+4.13	+2.15

It is interesting to note that the present value of each level annuity, on every basis, has been given in the form $f\bar{a}_x + g$, where f and g are functions of m (or constants), with $f \doteqdot 1$ and $g < 0$.

The paper increases one's awareness of lack of consistency in some of the formulas in use for net premiums and reserves. For example, Jordan's formula (p. 81),

$$\bar{P}(\bar{A}_x) \doteqdot \frac{(1+i/2)M_x}{N_x - \frac{1}{2}D_x},$$

combines Bases E and B. Consistency can, of course, be assured by using formulas containing just a single function, such as

$$\bar{P}(\bar{A}_x) = \frac{1}{\bar{a}_x} - \delta,$$

on any basis whatsoever.

The analysis of the direction of the error in the traditional mean reserves is interesting. The case of an increasing force of mortality is presented, with the direction of concavity of the graph of V_t in general determining the direction of the error. For the derivative of the force of mortality, I would suggest the addition of μ_x^2 to the form given. The results in Tables 7 and 8 which are thereby changed are given in Table III. These three tables all deal with the case $t \to 0^+$.

TABLE III

Age	1,000 μ_x'	1,000 K_2 (Table 7)	1,000 K_2 (Table 8)
15	.069	109.96	146.50
30	.049	110.00	164.24
45	.431	372.70	452.08
60	1.786	654.27	769.16
75	5.634	812.48	918.55

With the conclusions of the paper I concur wholeheartedly. Once again, may I express appreciation to Mr. Mereu for the significant addition to actuarial theory which his thorough, stimulating paper contributes.

(AUTHOR'S REVIEW OF DISCUSSION)

JOHN A. MEREU:

I would like to thank Mr. Crofts, Dr. Nesbitt, Mrs. Butcher and Mr. Sarason for their penetrating observations on my observations on actuarial approximations.

Mr. Crofts shows in his diagrams how the l_x curves underlying the various approximations can be compared graphically. Such a method is very appealing because of the ready manner in which it gives insight into the nature of an approximation. Although other functions besides l_x could be used as the gauge for comparing approximations they would not lend themselves to such a revealing graphic approach. All the possible l_x curves trace paths which have for any year of age the starting and end points in common.

The l_x curve defined by Basis S in the paper is one of a family of cubic curves with fixed beginning and end points and with predetermined slopes at those points. If the l_x curve so selected is to be realistic, satisfactory values of the initial and final slopes must of course be assigned. It is obvious that for the l-curve sketched by Mr. Crofts the assigned slope values were not intended to be realistic.

Mr. Crofts asks whether it is possible to conceive of a reserve function increasing indefinitely with a continually decreasing slope, assuming of course that there are no further premium payments. To answer this question the reserve can be equated to some single premium for level insurance. If the reserve exceeds the single premium for whole life insurance it will be equivalent to an endowment single premium for some period and if the reserve is less than the single premiums for whole life insurance it will be equivalent to a term single premium for some period. If we have the endowment situation it is clear that the reserve at maturity will, since it equals the face amount, exceed K_2 and therefore in the final phase of the period at least the curve will be increasing with increasing slope. If we have the term situation the reserve at expiry will, since it vanishes, be less than K_1, and therefore in the final phase of the period at least the curve will be decreasing. If we have the whole life situation the reserve and K_2 both exceed K_1 and approach the face amount, and the slope of the reserve curve at any point will depend on how the reserve compares with K_2. The following table compares K_2 and $\bar{A}_x$ on the CSO table for a num-

x	1,000 $\bar{A}_x$	1,000 K_2 (from Mrs. Butcher)
15	221.96	109.96
30	320.89	110.00
45	465.73	372.70
60	641.43	654.27
75	803.73	812.48

ber of ages. It is interesting to note that $\bar{A}_x$ exceeds K_2 throughout most of the range indicated. Thus the function $\bar{A}_x$ increases continuously but not always with increasing slope.

In the paper it was stated that if the reserve at the beginning of the year lies between K_1 and K_2, then unless further premiums are paid the reserve will reach a maximum and then decrease. This statement is not correct, since from the table above it is obviously possible for a curve which increases with decreasing slope to figuratively recover and increase with increasing slope. It would be interesting to have a comparison of K_2 and $\bar{A}_x$ made for some table subject to Gompertz's or Makeham's Law. I believe the above analysis answers Mr. Crofts' question except for the intriguing single premium whole life situation. The above analysis and that in the paper assumed level death and maturity benefits. Varying benefits have a material effect on the shape of the reserve curve.

Both Dr. Nesbitt and Mrs. Butcher discuss another basis of approximation (let us use Mrs. Butcher's notation and refer to Basis E) which assumes that both the decrements of discount and mortality are linear. The introduction of an element of approximation in the handling of the interest function makes an analysis of Basis E significantly different from that of Basis A to D respectively. The need for recognizing this difference becomes more apparent when we try to reconcile Dr. Nesbitt's relationship that $D^E_{x+t} > D^A_{x+t}$ with Mrs. Butcher's relationship that $\bar{A}^E_x > \bar{A}^A_x$. These two relationships are incompatible if we attempt to distinguish Basis E from Basis A by means of underlying l-curves alone.

It is necessary to consider that Basis E and Basis A make identical assumptions on the behavior of the l-curve between integral ages, and that they differ in the treatment of interest. Whereas Basis A just as the other bases discussed in the paper assumes a constant force of interest, Basis E in effect assumes a varying force of interest.

Basis E assumes that $P_0 = P_t(1 - dt)$, where P_t is the accumulation of P_0 with interest alone to time t. It can be shown that the force of interest

δ_t at time t under Basis E is given by $\delta_t^E = d/(1 - dt)$. A slightly different approach to the formula for $\bar{C}_x^E$ than that given by Dr. Nesbitt and Mrs. Butcher is then possible using the relationship:

$$\begin{aligned}
\bar{C}_x^E &= \int_0^1 D_{x+t}^E \mu_{x+t}^E \, dt \\
&= \int_0^1 [D_{x+t}^E (\mu_{x+t}^E + \delta_t^E) - D_{x+t}^E \delta_t^E] \, dt \\
&= \int_0^1 - [D_{x+t}'^E + D_{x+t}^E \delta_t^E] \, dt \\
&= D_x \int_0^1 \{ [d + q - 2dqt] - \delta_t^E [(1 - td)(1 - tq)] \} \, dt \\
&= D_x \cdot vq \left(1 + \frac{i}{2}\right) \\
&= C_x \left(1 + \frac{i}{2}\right).
\end{aligned}$$

From this we have the familiar formula $\bar{A}_x^E = A_x (1 + i/2)$.

Proceeding however to $a_x^{(m)E}$ and $\ddot{a}_x^E$ I must take exception with the formulas derived by Mrs. Butcher and Dr. Nesbitt. Their formulas are derived from $\bar{A}_x^E$ using the familiar $\bar{A} = 1 - \delta\bar{a}$ relationship. This relationship presupposes a constant force of interest which, of course, conflicts with initial hypothesis defining Basis E. Using the formula

$$\begin{aligned}
\bar{D}_x^E &= \int_0^1 D_{x+t}^E \, dt \\
&= D_x \left(\frac{1}{2} - \frac{d}{6}\right) + D_{x+1} \left(\frac{1}{2} + \frac{i}{6}\right),
\end{aligned}$$

I obtain the following expressions on Basis E:

$$\bar{a}_x^E = \ddot{a}_x \left(\frac{1}{2} - \frac{d}{6}\right) + a_x \left(\frac{1}{2} + \frac{i}{6}\right) = \left[1 + \frac{i - d}{6}\right] \ddot{a}_x - \left(\frac{1}{2} + \frac{i}{6}\right).$$

Similarly

$$\ddot{a}_x^{(m)E} = \left[1 + \frac{m^2 + 1}{6m^2}(i - d)\right] \ddot{a}_x - \left[\left(\frac{1}{2} - \frac{1}{2m}\right) + \frac{m^2 + 1}{6m^2} \cdot i\right].$$

Mrs. Butcher develops formulas for the force of mortality on the various assumptions. In some ways this function lends itself to being a common denominator better than the l-function. The l-curves have the property of sharing initial and final values. The μ-curves have the interesting property of subtending equal areas. This follows from the relationship

$$\int_0^1 \mu_{x+t}\,dt = \text{Colog}\; p_x = \text{Constant}\,.$$

By using the μ-curves as the common denominator of comparison it is possible to readily incorporate the relationships true for Gompertz and Makeham tables. The following interesting features are true of the derivatives of the μ-curves:

$$\mu'^{A}_{x+t} = [\mu^{A}_{x+t}]^2$$

$$\mu'^{B}_{x+t} = [\mu^{B}_{x+t} + \delta]^2$$

$$\mu'^{C}_{x+t} = -\;[\mu^{C}_{x+t}]^2$$

$$\mu'^{D}_{x+t} = -\;[\mu^{D}_{x+t} + \delta]^2\,.$$

Mrs. Butcher has remarked that my definition of $D_x^{(m)}$ appearing in the Appendix is not consistent with standard actuarial notation and that it leads to the incongruous result of $D_x = D_{x+1}$ if used for $m = 1$. It would certainly have been preferable if the standard definition had been used. I also agree that annuity formulas can be readily derived from corresponding insurance functions using the $A = 1 - d\ddot{a}$ and $IA = \ddot{a} - d(I\ddot{a})$ relationships. However, the independent derivations for the continuous functions at least did permit these relationships to be used for checking purposes.

Mrs. Butcher's Basis F is equivalent to assuming that all deaths in a year are concentrated at the mid-point of the year. Her extension of my Tables 2 and 4 to Bases E and F are appreciated. However, the values for 1,000 $\bar{a}_x^E$ if the formula above is accepted should appear as:

Age	1,000 $\bar{a}_x^E$	Error
15	26,323.44	+1.49
30	22,976.04	+1.06
45	18,075.62	+ .60
60	12,131.55	+ .94
75	6,644.45	+4.61

Mrs. Butcher has uncovered an error in my formula for μ_x'. The formula given in the paper is an approximation for $-l_x''$. She is correct in giving the formula as $\mu_x' = \mu_x^2 - l_x''$. The correction affects the values of μ_x' and of K_2 appearing in both Tables 7 and 8 of the paper. It should have been obvious that an error was developing in the paper on these tables, as one would expect μ_x' to be an increasing function. This is certainly true for a mortality table following Gompertz's or Makeham's Law.

Finally I would like to discuss briefly the nonmathematical discussion of the paper by Mr. Sarason. Mr. Sarason raises a question of semantics. Once an approximation has received statutory or official recognition, in some way it then in a manner of speaking becomes exact. It then follows that the theoretically exact formula or one with less theoretical error would be considered as an approximation relative to the official formula. Although such considerations as well as many others must be recognized in practice, they nevertheless do not disturb the underlying theories of actuarial science.