THE $\bar{M}$-LINEAR HYPOTHESIS AND VARYING INSURANCE

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Because mortality tables are generally tabulated at yearly intervals, actuaries frequently must make assumptions about intermediate values in developing formulas for insurance and annuity functions.

For example, by assuming a uniform distribution of deaths during each age interval one can develop the formula $\bar{A}_x = \frac{i}{\delta} A_x$.

Likewise, if one assumes that $D_{x+t}$ is linear in each age interval one can develop the formula $\bar{a}_x = a_x + \frac{1}{2}$.

I propose in this paper to examine the effect on insurance and annuity formulas of assuming $\bar{M}_{x+t}$ to be linear within each interval. Consider a varying insurance benefit $(\bar{V}\bar{A})_x$ issued at age x which pays a varying benefit $b_y$ if death occurs at age y.

The expected value of the benefit is given by

$$(\bar{V}\bar{A})_x = \int_0^\infty b_{x+t} \cdot v^t \cdot {}_tp_x \, \mu_{x+t} \, dt$$

$$= \sum_{k=0}^{\infty} \int_k^{k+1} b_{x+t} \cdot v^t \cdot {}_tp_x \, \mu_{x+t} \, dt$$

$$= \sum_{k=0}^{\infty} v^k \cdot {}_kp_x \cdot \int_k^{k+1} b_{x+t} \cdot v^{t-k} \cdot {}_{t-k}p_{x+k} \cdot \mu_{x+t} \, dt$$

$$= \sum_{k=0}^{\infty} v^k \, {}_kp_x \int_0^1 b_{x+k+s} \cdot v^s \cdot {}_sp_{x+k} \cdot \mu_{x+k+s} \, ds$$

$$= \sum_{k=0}^{\infty} {}_kE_x \cdot (\bar{V}\bar{A})^{\,1}_{x+k:\overline{1}|} \qquad (1)$$

We can, therefore, focus on a one year horizon and deal with $(\bar{V}\bar{A})^{\,1}_{y:\overline{1}|}$

Under the M-linear hypothesis

$$\bar{M}_{y+s} = \bar{M}_y + s \cdot \Delta\bar{M}_y \qquad 0 \le s \le 1 \qquad (2)$$

Taking [redacted] derivatives we get

$$-D_{y+s}\,\mu_{y+s} = \Delta\bar{M}_y = -\bar{C}_y$$

$$(\bar{V}\bar{A})^{1}_{y:\overline{1}|} = \int_0^1 b_{y+s} \cdot V^s \cdot {}_sp_y\,\mu_{y+s}\,ds$$

$$= \int_0^1 b_{y+s} \cdot \frac{D_{y+s}}{D_y}\,\mu_{y+s}\,ds$$

$$= \int_0^1 b_{y+s} \cdot \frac{\bar{C}_y}{D_y}\,ds = \bar{A}^{1}_{y:\overline{1}|} \cdot \int_0^1 b_{y+s}\,ds$$

$$= \bar{b}_y \cdot \bar{A}^{1}_{y:\overline{1}|}$$

where $\bar{b}_y = \int_0^1 b_{y+s}\,ds$ is the average death benefit for the year. You will note that under the $\bar{M}$-linear hypothesis varying insurances can be replaced in each year by an equivalent level amount payable at the moment of death. Using this property is familiar to actuaries. For example, Jordan advocates that $(\bar{I}\bar{A})_x \doteq (I\bar{A})_x - \frac{1}{2}\bar{A}_x$.

Having shown that M-linear has an appealing property I must now show what this assumption implies in terms of the survival and other functions.

From equation (3) we have

$$V^{y+s} \cdot l_{y+s} \cdot \mu_{y+s} = \bar{C}_y$$

$$\therefore \quad l_{y+s}\,\mu_{y+s} = (1+i)^{y+s} \cdot \bar{C}_y$$

$$\int_0^1 l_{y+s}\,\mu_{y+s}\,ds = \bar{C}_y \cdot \int_0^1 (1+i)^{y+s}\,ds$$

$$d_y = \bar{C}_y \cdot (1+i)^y \cdot \frac{i}{\delta}$$

$$\therefore \quad \bar{C}_y = \frac{\delta}{i} V^y d_y = \frac{\delta}{d} C_y$$

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Under UDD $\bar{C}_y^{UDD} = \frac{i}{\delta} C_y > \bar{C}_y^{\bar{M}}$

The M-linear results in a survival curve that travels above the linear or UDD curve but below the D-linear curve.

In the new text book on life contingencies the authors favour assumptions that are independent of interest. In particular they favour the linear or UDD assumption. Under this assumption

$$(\bar{V}\bar{A})^{1}_{y:\overline{1|}} = \int_0^1 b_t \, v^t \, {}_tp_y \, \mu_{y+t} \, dt$$

$$= \int_0^1 b_t \cdot v^t \, q_y \, dt$$

$$= v q_y \int_0^1 b_t \, (1+i)^{1-t} \, dt$$

$$= b^+ \cdot A^{1}_{y:\overline{1|}} \quad \text{where } b^+ = \int_0^1 b_t \, (1+i)^{1-t} \, dt$$

Under this assumption a varying amount payable at the moment of death during a year is equivalent to a level amount payable at the end of the year of death of an amount equal to the accumulation of the varying amount with interest to the end of the year.

The following table compares formulas for the $\bar{M}$-linear and UDD assumptions:

| Function | UDD | $\bar{M}$-Linear | D-Linear |
| --- | --- | --- | --- |
| $\bar{A}_x$ | $\frac{i}{\delta} A_x$ | $\frac{\delta}{d} A_x$ | $\frac{\delta}{d} A_x + \left(1 + \frac{\delta}{2} - \frac{\delta}{d}\right)$ |
| $\bar{a}_x$ | $\frac{id}{\delta^2} \ddot{a}_x - \frac{i-\delta}{\delta^2}$ | $\ddot{a}_x - \frac{\delta - d}{\delta d}$ | $\ddot{a}_x - \frac{1}{2}$ |
| $a(x)$ | $\frac{1}{2}$ | $\frac{\delta - d}{\delta d}$ | complex function of interest and $q_x$. |

Note: $a(x) = \dfrac{\int_0^1 t \, l_{x+t} \, \mu_{x+t} \, dt}{d_x}$ = average period lived in a year by those dying in the year.

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