Pricing of Options on Bonds by Binomial Lattices and by Diffusion Processes

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Abstract

This paper discusses the pricing of bond options and other interest-rate contingent claims by discrete- and continuous-time models. In the literature there are papers trying to adapt the binomial model developed for the pricing of stock options to the pricing of bond options. However, some of these adaptations are faulty as they do not eliminate riskless arbitrage opportunities. In this paper we present a binomial model which does not contain this error. We also present a continuous-time arbitrage-free model for the term structure of interest rates, with which one can determine option values.

I. Introduction

The option-pricing theory of F. Black and M. Scholes [Bl-S] has been described as the most important single advance in the theory of financial economics in the 1970's. These authors derive a formula for valuing a European call option on a non-dividend paying stock by showing that the option and stock can be combined linearly to form a riskless hedge. Since the appearance of the paper, there have been many attempts to extend its methodology to the determination of the values of options on default-free bonds and other interest-rate contingent claims.

The mathematics of the Black-Scholes theory has been simplified by Cox, Ross and Rubinstein [C-R-R] and Rendleman and Bartter [R-B1] under the assumption that stock-price movements can be described by a binomial lattice. Subsequently, Rendleman and Bartter [R-B2], Clancy [Cl] and Pitts [P1; P2] have tried to adapt the binomial model to price options on bonds. However, Bookstaber, Jacob and Langsam

[B-J-L, p. 19] write: "These problems cannot be corrected by wrapping baling wire around an existing model: taking the option models developed for equity options and simply changing the assets from stocks to bonds or to interest rates is a short cut that does not work."

In Section II we shall explain the objection of Bookstaber *et al.* In Section III we present a binomial model recently developed by Ho and Lee [H-L]. In Section IV we present a continuous-time arbitrage-free model for the term structure of interest rates, with which one can also determine option values.

II. Put-Call Parity and Riskless Arbitrages

An *option* is a contract entitling its holder to buy or sell a designated security at a specified price at a certain time or within a certain time period. The price that is paid for the security if the option is exercised is called the *striking price* or *exercise price.* The security is traded at the exercise price even if the price of the security has changed dramatically by the time when the option is exercised. The owner or holder of an option is under no obligation to buy or sell the security and will do so only if exercising the option is preferable to letting the option expire. (A financial *futures* contract, in contrast to an option contract, imposes a firm obligation on each party to buy or sell a predetermined position in the specified security.) A *call* option gives its owner the right to buy the security and a *put* option gives its owner the right to sell the security at the exercise price. The seller or writer of the call (or put) option has a contingent obligation to sell (or buy) if the option is exercised. In return for accepting the obligation, the seller receives an option premium from the buyer. The last day on which the option may be exercised is called the *expiration date* or *maturity date.* An *American* option is one that can be exercised at any time up to the maturity date. A *European* option is one that can be exercised only on the maturity date.

Let S(s) denote the value of a certain security at time s. For a fixed exercise price E, let the current value of the European call option and the European put option on the security, with a maturity date t time periods from now, be denoted by C_t and P_t, respectively. Let v_t denote the present value of 1 payable t time periods from now. (If the yield curve is flat, then

$$v_t = \exp\left(-\int_0^t \delta(s)\,ds\right) = \exp\left(-\int_0^t \delta\,ds\right) = e^{-\delta t} = v^t.)$$

Assuming that no dividend is paid on the security, we shall derive the identity

$$S(0) + P_t = C_t + v_t E, \qquad t \geq 0. \tag{2.1}$$

This formula, called the *put-call parity,* states that a position in a security (which pays no dividend) plus a European put option on the underlying security is equivalent to a European call option (with the same exercise price and expiration date) on the underlying security plus a pure discount bond maturing for the exercise price on the option exercise date. This relationship must be valid, regardless of the option-pricing model used.

To prove (2.1), note that, on exercise date, the left-hand side of (2.1) becomes

$$S(t) + Max\{E - S(t), 0\}, \tag{2.2}$$

while the right-hand side becomes

$$Max\{S(t) - E, 0\} + E. \tag{2.3}$$

Both (2.2) and (2.3) are equal to $Max\{S(t), E\}$. As there are no payments in the time interval $(0, t)$, identity (2.1) must hold. (Goodman [Go] examines the put-call parity for coupon-bearing instruments and its implications.)

If formula (2.1) does not hold for all t, there are riskless arbitrage opportunities. (An arbitrage is defined as the simultaneous purchase and sale of the same or equivalent security in order to profit from price discrepancies. An exposition on the no-arbitrage principle can be found in [Mi] and [In, chapter 2].) Unfortunately, in some of the models in the literature, the put-call parity formula need only hold for one period $(t = 1)$. For a numerical example showing how the put-call parity relationship may be violated in a multi-period setting, see [B-J-L, p. 5] or [Boo, p. 886].

Let us now give a condition that would eliminate such riskless arbitrages. Assume that, one time period from now, there can be n *states of the world,* each with a different term structure of interest rates. (In a binomial model, $n = 2$.) Corresponding to state i, $1 \leq i \leq n$, there exists a force-of-interest function δ_i which determines the values of all default-free bonds. For $t \geq 1$, consider a discount bond maturing for the value of 1 in t time periods from now, its present value being denoted by v_t. After one time period, the maturity time of the bond shortens by one and its value will be one of $v_{t-1, 1}$, $v_{t-1, 2}$, $v_{t-1, 3}$, ..., $v_{t-1, n}$, where

$$v_{t-1,i} = \exp\left(-\int_0^{t-1} \delta_i(s)\,ds\right).$$

Pictorially, we have
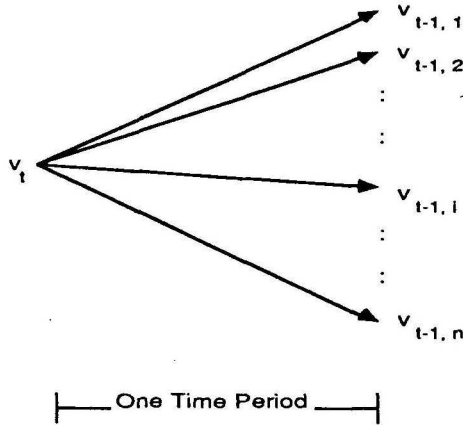
To eliminate riskless arbitrage opportunities in the model, we must have the inequalities

$$v_1 \operatorname*{Min}_i\{v_{t-1,i}\} \le v_t \le v_1 \operatorname*{Max}_i\{v_{t-1,i}\}, \qquad (2.4)$$

(where v_1 is the cost at time 0 of a discount bond maturing at time 1) or, more generally, for all sequences of real numbers $\{c_j\}$ and positive numbers $\{t_j \mid t_j \ge 1\}$,

$$v_1 \operatorname*{Min}_i\left\{\sum_j c_j v_{t_j-1,i}\right\} \le \sum_j c_j v_{t_j} \le v_1 \operatorname*{Max}_i\left\{\sum_j c_j v_{t_j-1,i}\right\}. \qquad (2.5)$$

(Buying a negative quantity of an asset is interpreted as shorting that asset.) As (2.5) should also hold for $\{-c_j\}$, it is sufficient to consider just one of the inequalities in (2.5), say,

$$\sum_j c_j v_{t_j} \le v_1 \operatorname*{Max}_i\left\{\sum_j c_j v_{t_j-1,i}\right\}. \qquad (2.5')$$

Suppose that (2.5') does not hold, i.e., there exist a sequence of real numbers $\{c_j\}$ and maturity dates $\{t_j \mid t_j \ge 1\}$ such that

$$\sum_j c_j v_{t_j} > v_1 \operatorname*{Max}_i\left\{\sum_j c_j v_{t_j-1,i}\right\}. \qquad (2.6)$$

Then a riskless arbitrage opportunity arises [B-J-L, p. 20]: Consider the portfolio comprised of selling at time 0, c_j units of discount bonds maturing at time t_j and using the

revenue (the left-hand side of (2.6)) to buy discount bonds maturing at time 1. The strategy is completed at time 1 by cashing the mature bonds and using the revenue to close out all positions. At time 1 and state i, the profit from this position is

$$\sum_j \frac{c_j v_{t_j}}{v_1} - \sum_j c_j v_{t_j-1,\,i},$$

which, regardless of i, is strictly positive. As the riskless profit is made with zero net investment, the rate of return is infinite.

In the language of *Functional Analysis*, we may rephrase (2.5') as

$$\phi(\mathbf{v}) \le \underset{i}{\text{Max}} \{\phi(\mathbf{v}_i)\}, \qquad\qquad (2.7)$$

for all (real-valued) linear functionals ϕ. We claim that inequality (2.7) holds if and only if $\mathbf{v}$ is in the convex hull of $\{\mathbf{v}_i\}$, i.e., there exist nonnegative numbers $\{\theta_i \mid 1 \le i \le n\}$ such that

$$\sum_i \theta_i = 1$$

and

$$\sum_i \theta_i \mathbf{v}_i = \mathbf{v}.$$

The "if" direction is obvious since the functionals ϕ are linear,

$$\phi(\mathbf{v}) = \phi\left(\sum_i \theta_i \mathbf{v}_i\right) = \sum_i \theta_i \phi(\mathbf{v}_i) \le \underset{i}{\text{Max}} \{\phi(\mathbf{v}_i)\}.$$

As there are only finitely many vectors $\{\mathbf{v}_i\}$ by assumption, their convex hull is compact. Thus, the "only if" direction follows from a *Separation Theorem for Convex Sets* [A-E, p. 1; Fr, section 1.6; Ti, p. 30; Val, Theorem 2.10]. (If we assume that there are only finitely many discount bonds, then the "only if" direction is an immediate consequence of the fact that every closed convex set in $\mathbb{R}^m$ is an intersection of half-spaces [S-W, Theorem 3.3.7].) Usually the separation theorems are proved via theorems of the Hahn-Banach type. For a proof of the "only if" direction by means of the Hahn-Banach Theorem and the equivalence between compactness and the finite intersection property, see the Appendix in [B-J-L].

Hence, we have shown that the riskless arbitrages arising from violating inequalities (2.5) are eliminated from the model if and only if there exist nonnegative

numbers $\{\theta_i \mid 1 \le i \le n\}$, independent of t, such that

$$\sum_i \theta_i = 1$$

and, for all t,

$$\frac{v_t}{v_1} = \sum_i \theta_i v_{t-1,i}. \tag{2.8}$$

These numbers may be called *arbitrage probabilities* [B-J-L, p. 13], *implied probabilities* [H-L, p. 1018] or *"risk-neutral" probabilities* [C-R-R, p. 235]. To eliminate all arbitrages we should require them to be strictly positive. We wish to stress that the number θ_i need not be the probability that state i will occur at time 1. However, if it turns out that, for each i, θ_i is the probability that state i will occur, then all bonds have the same expected return (i.e., no *term premiums* exist) and we say that the *local expectations hypothesis* holds [H-L, p. 1022].

The existence of these "probabilities" eliminates one-period arbitrage opportunities. How can multi-period arbitrages be eliminated in a discrete time and discrete state-space model? Using a "martingale" or "semigroup" argument, we can show that multi-period arbitrages do not exist if all one-period arbitrage opportunities are eliminated (cf. [Ga]).

We conclude this section by quoting [B-J-L, p. 17]: "Despite the pernicious nature of such lattice inconsistencies, it appears to have not been widely treated in the academic or professional literature. The potential for arbitrage-inconsistent lattices extends beyond the option pricing models to interest rate simulation methodology. Interest rate simulations are applied broadly for applications where the complexity of the option feature of financial instruments makes the usual option pricing models unworkable. ... Adjustable rate mortgages, CMOs (Collateralized Mortgage Obligations) and a number of financial products, such as the universal life programs and single premium deferred annuities, are typical candidates for simulation analysis. A simulation model that does not explicitly consider the full span of rates for the relevant portion of the yield curve and that is not founded on an arbitrage-free construction may not give dependable results for either pricing or exposure management."

III. Ho and Lee's Binomial Interest-Rate Movement Model

In this section we discuss a binomial model of term structure movement, recently proposed by Ho and Lee [H-L]. We now list the basic assumptions of the model; they are standard for discrete-time models of the perfect capital market. In the last section, we have already used some of these assumptions without stating them explicitly.

1.   The market is frictionless. There are no taxes, transaction costs or restrictions on short sales. All securities are perfectly divisible.

2.   The market clears at discrete points in time, which are separated in regular intervals. For simplicity, we use each period as a unit of time.

3.   The bond market is complete. There exist discount bonds for all maturities t, $t = 0, 1, 2, \dots$. (A discount bond of maturity t is a bond that pays 1 at time t, with no other payments to its holder.)

4.   At each time n, there are finitely many states of the world. For state i, we denote the equilibrium price of the discount bond of maturity t by $P(n, n+t, i)$. We require that, for all natural numbers n, t and i,

$$0 \le P(n, n + t, i) \le 1 \qquad (3.1)$$
$$P(n, n, i) = 1$$

and

$$P(n, \infty, i) = 0.$$

We note that the value of the second argument of the bond price function P must always be greater than or equal to the value of the first. In Section II above, we used the symbols v_t and $v_{t-1, i}$ for $P(0, t, 0)$ and $P(1, t, i)$, respectively.

At initial time, by convention, we have the 0-state. We assume that, at time 1, there are only two states of the world, denoted by 0 and 1.

Now, consider time 2. We have two choices. We may construct the model as

$P(0, t, 0)$

$P(2, t, 3)$

$P(2, t, 2)$

$P(2, t, 1)$

$P(2, t, 0)$
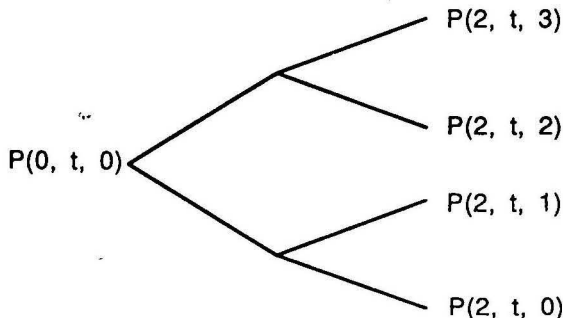
Then, as we continue the construction, we shall have 2^n states of the world at time n. Computationally, this will be cumbersome. Alternatively, we may construct the model as

P(0, t, 0)

P(2, t, 2)

P(2, t, 1)

P(2, t, 0)
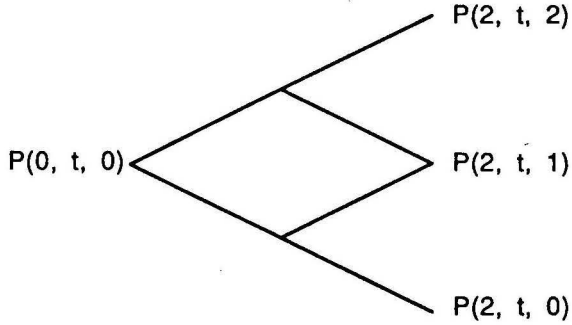
so that at time n we have only n+1 states, which are to be labelled with the natural numbers 0 to n. As this second model is much simpler to compute, we adopt this approach. Since we are labelling the states from 0 to n, we redraw the last figure as

P(2, t, 2)

P(2, t, 1)

P(0, t, 0)

P(2, t, 0)
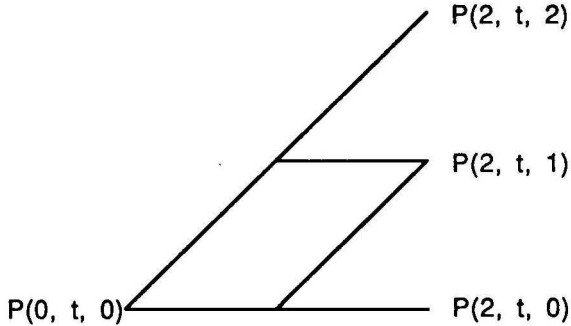
In general, we have the lattice:

States

10

8

4

2

0

0   2   4   6   8   10   Time
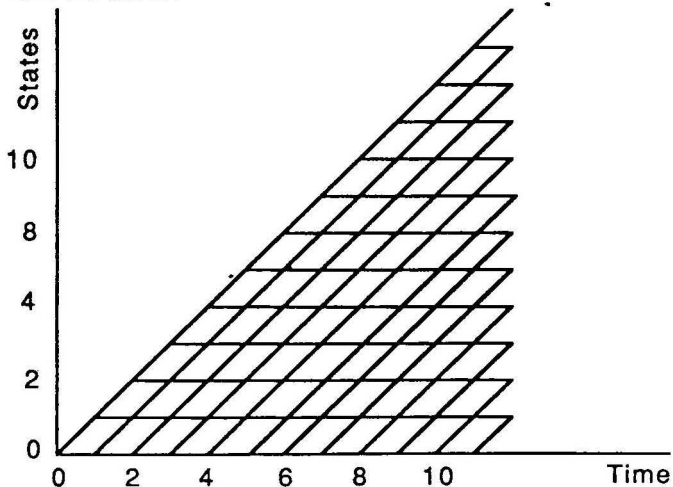
It is obvious from the last two pictures that there are two types of basic movements (as time passes) in this binomial lattice — *upward* (state i to state i+1) and *horizontal* (state i to state i). The term "horizontal movement" should not be taken to imply that the yield curve is to remain unchanged as time passes.

For the binomial-lattice model to be well defined, we need to impose a *path-independent* condition. The two values P(0, t, 0) and P(n, t, i) are "connected" by $\binom{n}{i}$ paths, each with i upward movements and n–i horizontal movements. All these paths must lead to the same value of P(n, t, i). For the model to be path independent, it is sufficient to assume that an upward movement followed by a horizontal movement is equivalent to a horizontal movement followed by an upward movement, i.e., starting with P(m, t, i), the two paths in the following figure give identical value for P(m+2, t, i), t ≥ m+2.
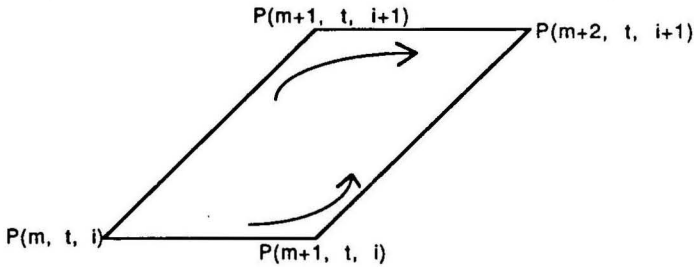
It is easy to see graphically how the "local" path-independent condition implies the "global" path-independent condition. (For an interesting discussion on path independence and portfolio insurance, see the prize-winning article [Ru].)

Let us now rewrite (2.8). It is required that, for each n and i, there exists a number θ(n, i) between 0 and 1 such that, for all t ≥ n+1,

$$\frac{P(n, t, i)}{P(n, n+1, i)} = \theta(n, i)P(n+1, t, i+1) + [1 - \theta(n, i)]P(n+1, t, i). \qquad (3.2)$$

As the bond prices are assumed to be uniquely defined at each node (n, i) of the binomial lattice, there exist two functions h and u such that, for all i, n and t (t ≥ n+1),

$$\frac{P(n, t, i)}{P(n, n+1, i)} h(n, t, i) = P(n+1, t, i) \qquad (3.3)$$

and

$$\frac{P(n, t, i)}{P(n, n+1, i)} u(n, t, i) = P(n+1, t, i+1) . \qquad (3.4)$$

(Note that h(n, n+1, i) = u(n, n+1, i) = 1.) Substituting the left-hand sides of (3.3) and (3.4) into the right-hand side of (3.2) yields

$$1 = \theta(n, i)u(n, t, i) + [1 - \theta(n, i)]h(n, t, i). \tag{3.5}$$

For simplicity we shall assume that $\theta(n, i) = \theta$, a constant. We shall also assume that the perturbation functions h and u depend only on the remaining time to maturity of the bond, i.e., there exist functions h and u such that

$$h(n, t, i) = h(t-n-1)$$

and

$$u(n, t, i) = u(t-n-1).$$

Hence, (3.2), (3.3), (3.4) and (3.5) may be simplified to

$$\frac{P(n, t, i)}{P(n, n+1, i)} = \theta P(n+1, t, i+1) + (1 - \theta)P(n+1, t, i), \tag{3.2'}$$

$$\frac{P(n, t, i)}{P(n, n+1, i)} h(t-n-1) = P(n+1, t, i), \tag{3.3'}$$

$$\frac{P(n, t, i)}{P(n, n+1, i)} u(t-n-1) = P(n+1, t, i+1) \tag{3.4'}$$

and

$$1 = \theta u(x) + (1 - \theta)h(x), \qquad x = 0, 1, 2, 3, \dots . \tag{3.5'}$$

(Ho and Lee [H-L] use the symbols π, h^* and h to denote our θ, h and u, respectively.)

Applying (3.3') and (3.4') to each other yields two expressions for $P(n+2, t, i+1)$:

$$\frac{P(n, t, i)}{P(n, n+1, i)P(n+1, n+2, i+1)} u(t-n-1)h(t-n-2) \tag{3.6}$$

and

$$\frac{P(n, t, i)}{P(n, n+1, i)P(n+1, n+2, i)} h(t-n-1)u(t-n-2) . \tag{3.7}$$

By the (local) path-independent condition, (3.6) and (3.7) are equal. Hence, for all positive integers x,

$$\frac{u(x)h(x-1)}{P(n+1, n+2, i+1)} = \frac{h(x)u(x-1)}{P(n+1, n+2, i)} , \tag{3.8}$$

from which we see that the ratio of ratios

$$\frac{h(x)}{h(x-1)} \Big/ \frac{u(x)}{u(x-1)} \tag{3.9}$$

is independent of x. Let the value of (3.9) be denoted by k. Then,

$$\frac{h(x)}{u(x)} = \frac{h(0)}{u(0)} k^x.$$

Since $u(0) = h(0) = 1$,

$$\frac{h(x)}{u(x)} = k^x. \tag{3.10}$$

Applying (3.5') to (3.10) yields

$$\frac{1 - \theta u(x)}{(1 - \theta)u(x)} = k^x.$$

Hence,

$$u(x) = \frac{1}{(1 - \theta)k^x + \theta} \tag{3.11}$$

and

$$h(x) = \frac{k^x}{(1 - \theta)k^x + \theta} = \frac{1}{(1 - \theta) + \theta k^{-x}}. \tag{3.12}$$

(In [H-L] the constant k is denoted by δ.)

Now, the value $P(n, t, i)$, $n \le t$, can be expressed in terms of θ, k and the initial bond prices $P(0, t, 0)$ and $P(0, n, 0)$. For $m < t$ and $m < s$,

$$\frac{P(m, t, j)u(t-m-1)}{P(m, s, j)u(s-m-1)} = \frac{P(m+1, t, j+1)}{P(m+1, s, j+1)} \tag{3.13}$$

by (3.4'). Similarly, by (3.3')

$$\frac{P(m, t, j)h(t-m-1)}{P(m, s, j)h(s-m-1)} = \frac{P(m+1, t, j)}{P(m+1, s, j)}. \tag{3.14}$$

Applying (3.13) i times and (3.14) $(n - i)$ times yields

$$\frac{P(0, t, 0)}{P(0, n, 0)} \frac{u(t-1)}{u(n-1)} \cdots \frac{u(t-i)}{u(n-i)} \frac{h(t-i-1)}{h(n-i-1)} \cdots \frac{h(t-n)}{h(0)} = \frac{P(n, t, i)}{P(n, n, i)}. \tag{3.15}$$

Since $h(0) = P(n, n, i) = 1$,

$$\frac{P(0, t, 0)}{P(0, n, 0)} \frac{u(t-1)}{u(n-1)} \cdots \frac{u(t-i)}{u(n-i)} \frac{h(t-i-1)}{h(n-i-1)} \cdots \frac{h(t-n+1)}{h(1)} h(t-n) = P(n, t, i). \tag{3.16}$$

By (3.10),

$$P(n, t, i) = \frac{P(0, t, 0)}{P(0, n, 0)} \frac{u(t-1)u(t-2) \cdots u(t-n)}{u(n-1)u(n-2) \cdots u(1)} k^{(t-n)(n-i)}$$

$$= \frac{P(0, t, 0)}{P(0, n, 0)} \frac{h(t-1)h(t-2) \cdots h(t-n)}{h(n-1)h(n-2) \cdots h(1)} k^{-(t-n)i} \tag{3.17}$$

Hence, the price $P(n, t, i)$, $n \le t$, can be expressed in terms of θ, k, $P(0, t, 0)$ and $P(0, n, 0)$. In the case of a one-period discount bond, we have the simple formula:

$$P(n, n+1, i) = \frac{P(0, n+1, 0)}{P(0, n, 0)} \frac{k^{n-i}}{(1-\theta)k^n + \theta}. \tag{3.18}$$

We are now ready to explain how a certain category of interest-rate contingent claims can be priced by the Ho-Lee binomial-lattice model. Examples of the contingent claims belonging to this category are callable and sinking-fund bonds, European and American bond options, interest rate futures (taking marking-to-market into account) and interest rate futures options.

Let C be an interest-rate contingent claim whose price $C(n, i)$ can be uniquely defined at each node (n, i) of the binomial lattice. Assume that C expires (or matures) at time T, with payoffs given by

$$C(T, i) = f(i), \quad i = 0, 1, 2, \dots, T. \tag{3.19}$$

Assume that the contingent claim pays $X(n, i)$ to its holder at time n and in state i, $1 \le n < T$, and satisfies its upper bound $U(n, i)$ and lower bound $L(n, i)$ conditions,

$$L(n, i) \le C(n, i) \le U(n, i). \tag{3.20}$$

If no arbitrage profit is to be realized in holding any portfolio of the contingent claim and the discount bonds, then

$$\frac{C(n, i)}{P(n, n+1, i)} = \theta[C(n+1; i+1) + X(n+1, i+1)] + (1-\theta)[C(n+1, i) + X(n+1, i)]. \tag{3.21}$$

Formula (3.21), a consequence of (3.2'), is called the *Risk-Neutral Pricing Formula*. A proof of it is given in Appendix B of [H-L]. It enables us to price the initial value of a contingent claim by the backward substitution procedure. The terminal condition (3.19) specifies the asset value in all states at time T. Then, formula (3.21) is used to determine the arbitrage-free price of the asset at one period before expiration. Let that price be $C^*(T-1, i)$. As the actual market price must satisfy boundary conditions (3.20), the market price must be

$$C(T-1, i) = Max\{L(T-1, i), Min[C^*(T-1, i), U(T-1, i)]\}, \quad 0 \le i \le T-1. \tag{3.22}$$

We now apply this procedure repeatedly, rolling back in time. That is, given the prices of the contingent claim in all states at time n, $\{C(n, i) \mid 0 \le i \le n\}$, we calculate the arbitrage-free prices of the contingent claim at time $n-1$, $\{C^*(n-1, i) \mid 0 \le i \le n-1\}$, by formula (3.21). Then, applying boundary conditions (3.20), we derive the market prices in all states at time $n-1$, i.e.,

$$C(n-1, i) = Max\{L(n-1, i), Min[C^*(n-1, i), U(n-1, i)]\}, \quad 0 \le i \le n-1.$$

After T steps, we reach the asset value at $n = 0$, and this is the initial price.

Now, suppose that the terminal condition (3.19) is given by $C(T, i) = 1$, $0 \le i \le T$, and that there are no other payments, i.e., $X(n, i) = 0$, and no boundary conditions. What is $C(n, i)$? Comparing (3.21) with (3.2'), we see that $C(n, i) = P(n, T, i)$. Thus, a discount bond can be viewed as a contingent claim, with terminal values $f(i) = 1$ and without any lower and upper bounds and interim cash flows.

It follows from (3.21) that in order to price interest-rate contingent claims, we only need to know θ and the prices for all one-year discount bonds $P(n, n+1, i)$ (which are computed by means of (3.18)). Hence, this model of asset pricing depends crucially on the stochastic movement of the short-term interest rate, and, for this reason, Ho and Lee call it a *one-factor model.*

Are there any problems with the Ho-Lee model? Yes, there are. The interest rates may become unreasonably high or even negative. Consider the $n+1$ possible one-period interest rates at time n; by (3.18) they are

$$\frac{1}{P(n, n+1, i)} - 1 = \frac{P(0, n, 0)}{P(0, n+1, 0)} \frac{(1 - \theta)k^n + \theta}{k^{n-i}} - 1, \qquad 0 \le i \le n. \qquad (3.23)$$

Without loss of generality, assume that k is less than one; hence $k^n \to 0$ as $n \to \infty$. For n large, formula (3.23) is approximately equal to

$$\frac{P(0, n, 0)\theta}{P(0, n+1, 0)k^n} k^i - 1, \qquad 0 \le i \le n.$$

From this we see that the interest rates may become unreasonably high or even negative. Furthermore, if we assume that the probability of an upward movement is p (and the probability of a horizontal movement is $1 - p$), then the probability that at time n the state of the world is i is given by

$$\binom{n}{i} p^i (1 - p)^{n-i}.$$

Hence, in the Ho-Lee model, the probability of negative or very high interest rates may not be insignificant.

Let us quote [B-J-L]: "The lognormal distribution conventionally used in option pricing models is inappropriate for modeling an interest rate process, since interest rates tend to vary within a fairly narrow band. ... Since in a lognormal process the variance grows linearly with time, a 20% standard deviation for rates over one year will imply a standard deviation of rates of 200% in 10 years. For example, if rates began at 10%, they would have a 5% chance of being above 50% at the end of 10 years. This is clearly at odds with historical experience. ... There is reason to think interest rates are mean

reverting, since abnormally high rates will lead to a shift in monetary policy to reduce rates while unusually low rates will lead to a less restrained policy which will lead rates to increase."

If we impose the conditions that interest rates should be nonnegative and mean reverting, then θ, h and u might need to be functions of n, t and i. This would complicate the model. In the next section we present a *continuous-time* arbitrage-free model of the term structure of interest rates, which excludes negative interest rates and has the mean-reverting property. To value an option in this model, we solve a parabolic partial differential equation subject to relevant boundary conditions.

IV. A Continuous-Time Arbitrage-Free Model of the Term Structure

The price at time s of a discount bond maturing (for the value of 1) at time t, $s \leq t$, determines a quantity $\delta(s, t)$ (called the force of interest by actuaries) such that the value $e^{-(t - s)\,\delta(s,\, t)}$ is the price of the bond. The instantaneous borrowing and lending rate at time s, called *spot* (or local) interest rate, is given by

$$r(s) \;=\; \delta(s, s) = \lim_{\substack{t \to s \\ t \geq s}} \delta(s, t). \tag{4.1}$$

We remark that in the finance literature the quantities $e^{\delta(s,\, t)} - 1$, $t \geq s$, are also called spot rates.

Following O. Vasicek [Vas, p. 179] we shall *assume* that the price of a discount bond is determined by the assessment, at time s, of the segment $\{r(\tau),\ s \leq \tau \leq t\}$ of the spot rate process over the term of the bond. Furthermore, we shall assume that the spot rate follows a continuous Markov process. The Markov property implies that the spot rate process is characterized by a single state variable, i.e., its current value. The probability distribution of the segment $\{r(\tau),\ s \leq \tau\}$ is thus completely determined by the value of r(s). Consequently, the force of interest of a discount bond is really a function of three variables, s, t and r(s), i.e., we should write $\delta(s, t, r(s))$ instead of $\delta(s, t)$. Recalling that in Section III the discount-bond price is denoted by P(s, t, i), where i designates a state of the world, we now define

$$P(s, t, r(s)) = e^{-(s - t)\,\delta(s,\, t,\, r(s))}.$$

Stochastic processes which are Markov and continuous are called *diffusion processes* and they can be described by a stochastic differential equation of the form

$$dr(s) = f(s, r(s))ds + \rho(s, r(s))dZ(s), \tag{4.2}$$

where $Z(s)$ is a Gauss-Wiener process with incremental variance ds, and $f(s, r(s))$ and $[\rho(s, r(s))]^2$ are the instantaneous drift and variance, respectively, of the process $r(s)$.

As $r(s)$ is stochastic, we need to use Itô's Lemma [In; Ma; M-B] to differentiate $P(s, t, r(s))$. Hence

$$dP = \frac{\partial P}{\partial s}ds + \frac{\partial P}{\partial r}dr + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}(dr)^2.$$

Since $(dr)^2 = \rho^2 ds +$ higher order terms in ds,

$$dP = \frac{\partial P}{\partial r}dr + [\frac{\partial P}{\partial s} + \frac{1}{2}\rho^2\frac{\partial^2 P}{\partial r^2}]ds. \tag{4.3}$$

Substituting dr from (4.2) into the first term of the right-hand side of (4.3) yields

$$dP = [\frac{\partial P}{\partial s} + f\frac{\partial P}{\partial r} + \frac{1}{2}\rho^2\frac{\partial^2 P}{\partial r^2}]ds + \rho\frac{\partial P}{\partial r}dZ. \tag{4.4}$$

With the definitions

$$\mu(s, t, r(s)) = [\frac{\partial P}{\partial s} + f\frac{\partial P}{\partial r} + \frac{1}{2}\rho^2\frac{\partial^2 P}{\partial r^2}]\frac{1}{P} \tag{4.5}$$

and

$$\sigma(s, t, r(s)) = -\frac{\rho}{P}\frac{\partial P}{\partial r}, \tag{4.6}$$

formula (4.4) can be written as

$$\frac{dP}{P} = \mu ds - \sigma dZ. \tag{4.7}$$

There is a negative sign in definition (4.6) because the partial derivative P_r is expected to be negative.

Now, consider an investment portfolio consisting of two discount bonds of different maturity dates. Assume that proportion α is invested in the first bond which matures at time t_1 and proportion $(1 - \alpha)$ is invested in the second bond which matures at time t_2. (The number α is not restricted between zero and one.) For $i = 1, 2$, write

$$\mu_i = \mu(s, t_i, r(s))$$

and

$$\sigma_i = \sigma(s, t_i, r(s)).$$

Let W denote the value of the portfolio. It follows from (4.7) that

$$\frac{dW}{W} = [\alpha\mu_1 + (1-\alpha)\mu_2]ds - [\alpha\sigma_1 + (1-\alpha)\sigma_2]dZ. \tag{4.8}$$

If α is chosen such that

$$\alpha\sigma_1 + (1-\alpha)\sigma_2 = 0, \tag{4.9}$$

then the portfolio has no risk. As a risk-free portfolio can only earn at the risk-free rate, we have

$$\alpha\mu_1 + (1-\alpha)\mu_2 = r(s). \tag{4.10}$$

(To maintain the portfolio risk-free as time passes, we need to keep changing α in accordance to (4.9).) Substituting the expression for α from (4.9) into (4.10) yields

$$\frac{\mu_1 - r(s)}{\sigma_1} = \frac{\mu_2 - r(s)}{\sigma_2}. \tag{4.11}$$

Since (4.11) is valid for arbitrary maturity dates t_1 and t_2, the ratio

$$\frac{\mu(s, t, r(s)) - r(s)}{\sigma(s, t, r(s))}$$

is independent of t. Let $q(s, r)$ denote the common value of such ratio for a discount bond of any maturity date, i.e., given that the current spot rate is $r(s) = r$,

$$q(s, r) = \frac{\mu(s, t, r) - r}{\sigma(s, t, r)}, \qquad s \le t. \tag{4.12}$$

Vasicek [Vas] calls the quantity $q(s, r)$ the *market price of risk,* as it specifies the increase in expected instantaneous rate of return on a bond per an additional unit of risk. It seems obvious that q should be nonnegative.

$$\mu = r + q\sigma$$
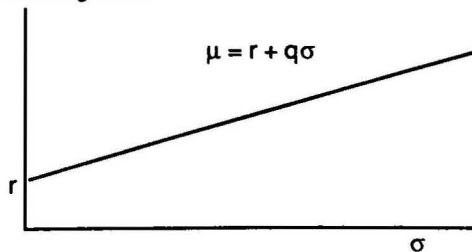
Substituting the expressions for μ and σ from (4.5) and (4.6) into (4.12) and rearranging, we obtain the *term structure equation*

$$\frac{\partial P}{\partial s} = rP - (f + \rho q)\frac{\partial P}{\partial r} - \frac{\rho^2}{2}\frac{\partial^2 P}{\partial r^2}, \qquad s \le t. \qquad (4.13)$$

The term structure equation is a parabolic partial differential equation for $P(s, t, r(s))$. Once the character of the spot rate process $r(s)$ is described and the market price of risk $q(s, r)$ specified, the bond prices are obtained by solving (4.13) subject to the boundary condition

$$P(s, s, r) = 1. \qquad (4.14)$$

The term structure of interest rates $\delta(s, s + T)$, $T \ge 0$, is then calculated from the equation

$$\delta(s, s + T) = -\frac{\ln P(s, s + T, r(s))}{T}. \qquad (4.15)$$

We remark that, for a bond paying coupons continuously at rate C per unit time, we add the term C to the right-hand side of (4.13). Also, by imposing appropriate boundary conditions, we can use (4.13) to evaluate savings bonds, retractable bonds and callable bonds [B-S1]. (Numerical schemes for solving equations such as (4.13) are presented in [B-S2; C2; In].) For example, the price of a callable bond, $P_C(s, t, r(s))$, is subject to the boundary condition which is specified for the call feature:

$$P_C(s, t, r(s)) \le E(s), \qquad s \le t,$$

where $E(s)$ is the exercise price at time s. The value of the call option, $C(s, t, r(s))$, can be derived as the difference between the prices of noncallable and callable bonds,

$$C(s, t, r(s)) = P(s, t, r(s)) - P_C(s, t, r(s)).$$

To complete the term-structure model, we need to specify the instantaneous drift $f(s, r)$ and standard deviation $\rho(s, r)$ in (4.2) and the market price of risk $q(s, r)$ defined by (4.12). As interest rates should be nonnegative, we impose the condition $f(s, 0) \ge 0$ and $\rho(s, 0) = 0$, for all s, so that

$$dr\big|_{r=0} \ge 0,$$

i.e., there is a natural absorbing or reflecting barrier at $r = 0$. We remark that setting $r = 0$ in (4.13) now yields the boundary condition

$$\frac{\partial P}{\partial s} = -f(s, 0)\frac{\partial P}{\partial r}, \qquad s \le t. \qquad (4.14a)$$

For simplicity we shall assume that the functions f, ρ and q do not depend on the time

variable s explicitly, i.e., $f(s, r) = f(r)$, $\rho(s, r) = \rho(r)$ and $q(s, r) = q(r)$.

Consider

$$f(r) = \kappa(\gamma - r), \tag{4.16}$$

where κ and γ are positive constants. The current value of a spot rate process with such a drift term is pulled towards a long-run (or steady-state) mean γ with a speed proportional to its difference from the mean. Definition (4.16) requires that the spot rate follows a first-order autoregressive process (elastic random walk).

Since interest rate fluctuations have generally been greater in periods of high interest rates and we require the condition $\rho(0) = 0$, we shall assume that $\rho(r)$ is proportional to a positive power of r. Following Cox, Ingersoll and Ross [C-I-R2], we let $\rho(r)$ be proportional to the square root of r. Thus, the spot rate process is governed by the stochastic differential equation:
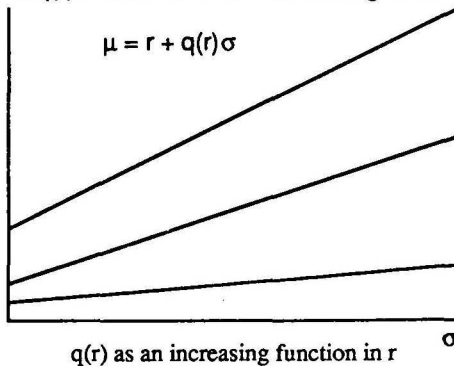
$$dr(s) = \kappa[\gamma - r(s)]ds + \upsilon\sqrt{r(s)}\, dZ(s), \tag{4.17}$$

where κ, γ and υ are positive constants. (A discussion on how to estimate the parameters of a diffusion process similar to (4.17) using the method of maximum likelihood can be found in the Appendix of [Og].)

Now, (4.13) becomes

$$\frac{\partial P}{\partial s} = rP - [\kappa(\gamma - r) + \upsilon\sqrt{r}\, q]\frac{\partial P}{\partial r} - \frac{\upsilon^2 r}{2}\frac{\partial^2 P}{\partial r^2}. \tag{4.18}$$

It seems reasonable that $q(r)$ should be a nondecreasing function in r.

$$\mu = r + q(r)\sigma$$

q(r) as an increasing function in r σ

Let us choose the market price of risk q to be of the form

$$q(s, r) = k\sqrt{r}, \quad \text{k a nonnegative constant,}$$

so that the partial differential equation (4.18) will not be too complicated. With the

definition

$$\lambda = -\upsilon k,$$

(4.18) becomes

$$\frac{\partial}{\partial s}P = rP + [\lambda r - \kappa(\gamma - r)]\frac{\partial}{\partial r}P - \frac{\upsilon^2 r}{2}\frac{\partial^2}{\partial r^2}P. \tag{4.19}$$

Cox, Ingersoll and Ross [C-I-R2, p. 393] call λ the *market risk parameter*. They derive (4.19) in the context of a sophisticated intertemporal general equilibrium asset pricing model [C-I-R2, (22)].

We must point out that not every choice of q leads to a consistent model. For instance, if we choose

$$q(s, r) = \frac{k_1 + k_2 r}{\sqrt{r}}, \qquad\qquad k_1, k_2 \text{ constants}, \qquad k_1 \neq 0,$$

we get a model guaranteeing arbitrage opportunities [C-I-R2, p. 398]. (Note that q has a singularity at $r = 0$.) To see the inconsistency, let us start with (4.7):

$$dP = \mu P ds - \sigma P dZ$$

$$= (r + q\sigma)P ds - \sigma P dZ$$

$$= (rP + q\sigma P)ds - \sigma P dZ$$

$$= (rP - q\rho P_r)ds + \rho P_r dZ$$

$$= (rP - q\upsilon\sqrt{r}P_r)ds + \upsilon\sqrt{r}P_r dZ$$

$$= [rP - (k_1 + k_2 r)\upsilon P_r]ds + \upsilon\sqrt{r}P_r dZ \cdot$$

$$= -k_1\upsilon P_r ds, \quad \text{if } r = 0.$$

The disappearance of the dZ term means that the bond's return over the next instant is riskless. As the risk-free rate r is set to be zero, we must have $k_1\upsilon P_r = 0$, or $P_r = 0$. However, formula (4.29) below (with appropriate interpretation of the constants a and b) shows that $P_r \neq 0$ when $r = 0$.

We now solve (4.19) under the boundary condition $P(s, s, r) = 1$. Define $x = t - s$ (the time to maturity), $a = \kappa\gamma$, $b = \lambda + \kappa$ and $c = \upsilon^2/2$. Put $\Pi(x, r) = P(s, t, r)$. Then, (4.19) becomes

$$-\Pi_x = r\Pi + (-a + br)\Pi_r - cr\Pi_{rr}. \tag{4.20}$$

The boundary condition is $\Pi(0, r) = 1$. We guess that the solution is of the form

$$\Pi(x, r) = \exp[A(x) - B(x)r], \tag{4.21}$$

with $A(0) = B(0) = 0$ [In, p. 397]. Substituting (4.21) into (4.20) yields

$$-[A'(x) - B'(x)r]\Pi = r\Pi + (a - br)B(x)\Pi - cr[B(x)]^2\Pi,$$

from which we get a pair of equations in the variable x only:

$$-A'(x) = aB(x) \tag{4.22}$$

and

$$B'(x) = 1 - bB(x) - c[B(x)]^2. \tag{4.23}$$

Now, for $\beta^2 > 4\alpha\gamma$,

$$\int \frac{d\zeta}{\alpha\zeta^2 + \beta\zeta + \gamma} = (\beta^2 - 4\alpha\gamma)^{-1/2} \ln \left| \frac{2\alpha\zeta + \beta - (\beta^2 - 4\alpha\gamma)^{1/2}}{2\alpha\zeta + \beta + (\beta^2 - 4\alpha\gamma)^{1/2}} \right|. \tag{4.24}$$

Put

$$w = \sqrt{b^2 + 4c} = \sqrt{(\lambda + \kappa)^2 + 2\upsilon^2}. \tag{4.25}$$

Applying (4.24) to (4.23) yields

$$-wx + \text{constant} = \ln \left| \frac{2cB(x) + b - w}{2cB(x) + b + w} \right|.$$

Using the condition $B(0) = 0$, we have

$$\frac{b - w}{b + w} e^{-wx} = \frac{2cB(x) + b - w}{2cB(x) + b + w},$$

or

$$\begin{aligned} B(x) &= \frac{2(1 - e^{-wx})}{b(1 - e^{-wx}) + w(1 + e^{-wx})} \\ &= \frac{2(e^{wx} - 1)}{b(e^{wx} - 1) + w(e^{wx} + 1)}. \end{aligned} \tag{4.26}$$

It follows from (4.22) and the condition $A(0) = 0$ that

$$A(x) = -a\int_0^x B(s)\, ds.$$

Let $d(x)$ denote the denominator in (4.26),

$$d(x) = (b - w)(e^{wx} - 1) + 2we^{wx} \tag{4.27}$$

$$= (b + w)(e^{wx} - 1) + 2w. \tag{4.28}$$

Then,

$$B(x) = \frac{2}{b - w} \frac{(b - w)(e^{wx} - 1)}{d(x)}$$

$$= \frac{2}{b-w}[1 - \frac{2}{b+w}\frac{d'(x)}{d(x)}] . \qquad \text{by (4.28)}$$

Integrating the last expression from 0 to x yields

$$A(x) = \frac{-2a}{b-w}[x - \frac{2}{b+w}\ln\frac{d(x)}{d(0)}]$$

$$= \frac{-2a}{b-w}[x - \frac{2}{b+w}\ln\frac{d(x)}{2w}].$$

As $(w-b)(w+b) = 2\upsilon^2$,

$$A(x) = \frac{2a}{\upsilon^2}[\frac{(b+w)x}{2} + \ln\frac{2w}{d(x)}].$$

Hence,

$$\Pi(x, r) = \exp[A(x) - B(x)r]$$

$$= \left[\frac{2we^{(b+w)x/2}}{d(x)}\right]^{2a/\upsilon^2} \exp\left[\frac{-2r(e^{wx} - 1)}{d(x)}\right]. \qquad (4.29)$$

Adapting to our notation, we now quote [C-I-R2, pp. 393-394]: "The bond price is a decreasing convex function of the interest rate and an increasing (decreasing) function of time (maturity). ... The bond price is a decreasing convex function of the mean interest rate level γ and an increasing concave (decreasing convex) function of the speed of adjustment parameter κ if the interest rate is greater (less) than γ. ... Bond prices are an increasing concave function of the market risk parameter λ. ... The bond price is an increasing concave function of the interest rate variance υ^2. ... For the discount bonds we are now considering, the yield to maturity, $\delta(s, t, r(s))$, is defined by

$$\exp[-(t - s)\delta(s, t, r(s))] \equiv P(s, t, r(s)).$$

As maturity nears, the yield to maturity approaches the current interest rate independently of any of the parameters. As we consider longer and longer maturities, the yield approaches a limit which is independent of the current interest rate:

$$\delta(s, \infty, r) = \frac{2\kappa\gamma}{w + \kappa + \lambda} .$$

With an interest rate in excess of $\kappa\gamma/(\kappa + \lambda)$, the term structure is falling. For intermediate values of the interest rate, the yield curve is humped. ... An increase in the current interest rate increases yields for all maturities, but the effect is greater for shorter maturities.

Similarly, an increase in the steady state mean γ increases all yields, but here the effect is greater for longer maturities. The yields to maturity decrease as υ^2 or λ increases, while the effect of a change in κ may be of either sign depending on the current interest rate."

V. Concluding Remarks

From the last paragraph, we see that the mean-reverting square-root spot-rate process provides "realistic" term structures of interest rates. The yield curve is rising for low spot rates and falling for high spot rates. There is a range of spot rates which produces a humped yield curve. The volatility of "long" rates decreases with maturity. Interest rates are never negative. The variance of the interest rate is an increasing function of the interest rate. These are some reasonable properties of interest rates. However, in this diffusion process model, the long rates are completely determined by the level of the spot rate. As Brennan and Schwartz [B-S3] point out, such single-parameter models are unlikely to be able to reproduce observed yield curves.

On the other hand, Ho and Lee [H-L] take the initial term structure as exogenously given and determine the short-rate movement to price the contingent claims. When the discount bonds, viewed as contingent claims, are priced by this term structure movement, they fit the initial term structure. Therefore, the term structure movement in their model is not used to determine the equilibrium term structure. Rather, it is assured to be consistent with the initial term structure and is used to price other contingent claims. The model does not impose any condition on the initial term structure. Besides the ease of computation, this is another elegant feature of their binomial model.

We should mention that Brennan and Schwartz [B-S3] and Richard [Ri] have developed diffusion models of interest rates with two state variables. For an excellent survey of these more sophisticated models, see [Sh, sec. 3].

Redington's theory of immunization [Re] can be extended to stochastic models of the term structure of interest rates. As the topic is beyond the scope of this paper, we refer the interested reader to [Al; Bo1; Bo2; C-I-R1; K-B-T; G-R].

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