QFI QF Model Solutions
Spring 2024

1. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1b) Understand the importance of the no-arbitrage condition in asset pricing.

Sources:
Hirsa, Ali and Neftci, Salih N., 3rd Edition - Ch. 2

Commentary on Question:
This question tests candidates' understanding of arbitrage with a simple one-period model.

Solution:

(a) Determine the range of r such that there are no arbitrage possibilities.

Commentary on Question:
Most candidates were able to derive an accurate range of the risk-free rate. A few candidates treated the rate as an annual rate and obtained incorrect ranges.

Let Q_u and Q_d be the risk-neutral probabilities that the security will go up and go down after 6 months, respectively. Then by the arbitrage theorem, we have

$$Q_u + Q_d = 1$$
$$Q_u \frac{120}{1+r} + Q_d \frac{60}{1+r} = 100$$

From the above equations, we can get

$$60Q_u + 60 = 100(1+r)$$

Since $0 \leq Q_u \leq 1$, we have

$$60 \leq 100(1+r) \leq 120$$

which gives the following no-arbtrage range of r

$$-0.4 \leq r \leq 0.2$$

1. Continued

Alternatively:

$$0 \le Q_u = \frac{1+r-d}{u-d} = \frac{0.4+r}{0.6} \le 1 \;\Rightarrow\; -0.4 \le r \le 0.2.$$

(b) Calculate and interpret the state prices.

Commentary on Question:
Most candidates were able to obtain and interpret the state prices correctly.

Let ψ_u and ψ_d denote the state prices corresponding to the up state and the down state, respectively. Then by the arbitrage theorem, we have

$$1 = (1+0.06)\psi_u + (1+0.06)\psi_d$$
$$100 = 120\psi_u + 60\psi_d$$

Solving the above equations gives the state prices:

$$\psi_u = 0.7233$$
$$\psi_d = 0.2201$$

The state prices can be interpreted as follows:

1. ψ_u is the price investors are willing to pay for an insurance policy that pays 1 in the up state and nothing in the down state.
2. ψ_d is the price investors are willing to pay for an insurance policy that pays 1 in the down state and nothing in the up state.

(c) Calculate the no-arbitrage price of a European call option with strike price of 100 that expires in 6 months.

Commentary on Question:
Most candidates obtained the correct price. A few candidates used a wrong risk-free rate.

Let Q_u and Q_d be the risk-neutral probabilities that the security will go up and go down after 6 months, respectively. Let C be the no-arbitrage price of the option. Since $r = 0.06$, the arbitrage theorem gives

$$Q_u + Q_d = 1$$
$$Q_u \frac{120}{1.06} + Q_d \frac{60}{1.06} = 100$$

1. Continued

$$Q_u \frac{(120-100)^+}{1.06} + Q_d \frac{(60-100)^+}{1.06} = C$$

Solving the first two equations gives

$$Q_u = 0.7667$$
$$Q_d = 0.2333$$

Plugging the risk-neutral probabilities into the third equation, we get

$$C = 0.7667 \times \frac{20}{1.06} = 14.46$$

Alternatively:

$$Q_u = \frac{1+r-d}{u-d} = \frac{0.46}{0.6} = 0.7667$$

(d) Describe two general situations in which arbitrage opportunities can arise.

Commentary on Question:
Most candidates gave correct cases when arbitrage opportunities occur.

Arbitrage opportunities can arise in two different fashions:
One can make a series of investments with no current net commitment, yet expect to make a positive profit.

A portfolio can ensure a negative net commitment today, while yielding nonnegative profits in the future.

(e) Construct a replicating portfolio and use it to price the derivative.

Commentary on Question:
Many candidates did not obtain the replicating portfolio correctly. Some candidates used options to replicate the derivative.

Let x be the number of shares of the security and let y be the money deposited/borrowed. The portfolio should replicate the payoffs of the derivative in both the up and the down states.

$$120x + 1.06y = 22$$
$$60x + 1.06y = 10$$

1. Continued

Solving the equations gives

$$x = 0.2$$
$$y = -1.8868$$

That is, the replicating portfolio consists of 0.2 shares of the security and 1.8868 is borrowed. The total value of the replicating portfolio at time 0 is

$$100 \times 0.2 - 1.8868 = 18.1132$$

Hence the value of the derivative at time 0 should be 18.1132.

2. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

3. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

(1j) Understand and apply Girsanov's theorem in changing measures.

(3a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

Sources:

Problems and Solutions in Mathematical Finance: Stochastic Calculus, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014

An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., Third Edition, Second Printing 2014

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Derive the closed-form expression for $S(t)$ using Ito's Lemma.

Commentary on Question:

Candidates performed well on this part. It is a straightforward application of Ito's Lemma to a geometric Brownian motion with constant drift and volatility.

$$F(t) = \log S(t)$$

$$dF = \frac{1}{S}dS + 0 * dt - \frac{1}{2S^2}(dS)^2$$

$$= \left(r - \frac{\sigma^2}{2}\right)dt + \sigma dW(t)$$

2. Continued

By integrating both sides, we find:

$$F(t) - F(0) = \left(r - \frac{\sigma^2}{2}\right) t + \sigma W(t)$$

$$\log \frac{S(t)}{S(0)} = \left(r - \frac{\sigma^2}{2}\right) t + \sigma W(t)$$

$$S(t) = S(0) e^{\left(r - \frac{\sigma^2}{2}\right) t + \sigma W(t)}$$

(b)

(i) Derive the stochastic dynamics of $Y_n(t)$ in the form

$$dY_n(t) = \mu\big(t, S(t)\big)dt + \sigma\big(t, S(t)\big)dW(t)$$

(ii) Show that $E_t^{\mathbb{Q}}\big(Y_1(T)\big) = Y_1(t)$.

(iii) Prove that $Y_n(t)$ is a sub-martingale for $n \geq 2$ and $t < \infty$.

Commentary on Question:
Candidates performed as expected on this section. Most candidates were able to properly apply Ito's Lemma again to the new stochastic process. For part (iii), many candidates proved the main property of a sub-martingale regarding the drift of the Expectation, but not the other defining properties.

(i)

$$\begin{aligned} dY_n(t) = &-re^{-rt}S^n(t)dt + ne^{-rt}S^{n-1}(t)dS(t) \\ &+ \frac{1}{2}n(n-1)e^{-rt}S^{n-2}(t)\big(dS(t)\big)^2 \end{aligned}$$

Substituting $dS(t) = rS(t)dt + \sigma S(t)dW(t)$ and $\big(dS(t)\big)^2 = \sigma^2 S^2(t)dt$, we find

$$\begin{aligned} dY_n(t) &= e^{-rt}S^n(t)\left[-r + nr + \frac{1}{2}n(n-1)\sigma^2\right]dt + n\sigma e^{-rt}S^n(t)dW(t) \\ &= e^{-rt}S^n(t)\left[(n-1)\left(r + \frac{1}{2}n\sigma^2\right)\right]dt + n\sigma e^{-rt}S^n(t)dW(t) \end{aligned}$$

(ii) When n = 1, we see the stochastic dynamics of the process simplifies to:

$$\begin{aligned} dY_1(t) = \; &e^{-rt}S^n(t)\left[(1-1)\left(r + \frac{1}{2}n(1)\sigma^2\right)\right]dt + (1)\sigma e^{-rt}S^n(t)dW(t) \\ &= 0dt + \; \sigma e^{-rt}S^n(t)dW(t) \\ &= \; \sigma e^{-rt}S^n(t)dW(t) \end{aligned}$$

2. **Continued**

Therefore, $E_t^{\mathbb{Q}}[Y_1(T) - Y_1(t)] = E_t^{\mathbb{Q}}\left[\int_t^T \sigma e^{-rt} S^n(u) dW(u)\right]$

The expression on the right is an Ito integral since the integrand is adapted to the same filtration, making its expectation 0.

$$E_t^{\mathbb{Q}}[Y_1(T) - Y_1(t)] = 0$$
$$E_t^{\mathbb{Q}}[Y_1(T)] = E_t^{\mathbb{Q}}[Y_1(t)]$$
$$E_t^{\mathbb{Q}}[Y_1(T)] = Y_1(t)$$

Since $Y_1(t)$ is $\mathcal{F}_t$-measurable.

(iii)
$Y_n(t)$ is clearly $\mathcal{F}_t$-adapted.

Next, we should check if it has bounded variation, i.e. is $E_0^{\mathbb{Q}}(|Y_n(t)|) < \infty$.

$$E_0^{\mathbb{Q}}(|Y_n(t)|) = E_0^{\mathbb{Q}}\left(\left|e^{-rt} S^n(0) e^{\left(r - \frac{\sigma^2}{2}\right)nt + \sigma n W(t)}\right|\right)$$
$$= S^n(0) e^{\left(r - \frac{\sigma^2}{2}\right)nt - rt + \frac{\sigma^2 n^2}{2}t} < \infty$$

Given that $t < T$, i.e. finite, the above expression must be finite.

Lastly, we will show the key sub-martingale property regarding expected drift.

In part (bi), we found that the drift of $Y_n(t)$ was $e^{-rt} S^n(t)\left[(n-1)\left(r + \frac{1}{2}n\sigma^2\right)\right]$

For $n \geq 2$, the expression in the brackets is always positive, making

$$E_t^{\mathbb{Q}}\left(\int_t^T \mu\big(u, S(u)\big) du\right) \geq 0$$

Therefore, $E_t^{\mathbb{Q}}(Y_n(T)) \geq Y_n(t)$.

Given the 3 properties above are satisfied, $Y_n(t)$ is a sub-martingale.

2. **Continued**

(c)

(i) Determine, using Girsanov theorem, an equivalent martingale measure $\mathbb{Q}^A$ such that $Y_n(t)$ is a martingale with respect to $\mathbb{Q}^A$.

(ii) Verify your result by deriving the dynamics of $Y_n(t)$ under $\mathbb{Q}^A$.

Commentary on Question:
Candidates performed reasonably well on this section.

Using Girsanov's theorem, since $W(t)$ is a standard Brownian motion in the probability space, an equivalent martingale measure $\mathbb{Q}^A$ exists with a different standard Brownian motion, $\widetilde{W}(t)$, such that $\widetilde{W}(t) = W(t) + \int \gamma(u)du$.

Let $\gamma(t) = \frac{\left((n-1)\left(r+\frac{1}{2}\sigma^2 n\right)\right)}{\sigma n}$ so that

$$d\widetilde{W}(t) = dW(t) + \frac{\left((n-1)\left(r+\frac{1}{2}\sigma^2 n\right)\right)}{\sigma n}dt$$

Therefore, under $\mathbb{Q}^A$, the dynamics of $Y_n(t)$ are

$$\begin{aligned}
dY_n(t) &= Y_n(t)\left[(n-1)\left(r+\frac{1}{2}n\sigma^2\right)\right]dt + n\sigma Y_n(t)dW(t) \\
&= Y_n(t)\left[(n-1)\left(r+\frac{1}{2}n\sigma^2\right)\right]dt \\
&\quad + n\sigma Y_n(t)\left[d\widetilde{W}(t) - \frac{\left((n-1)\left(r+\frac{1}{2}\sigma^2 n\right)\right)}{\sigma n}dt\right] \\
&= Y_n(t)\left[(n-1)\left(r+\frac{1}{2}n\sigma^2\right)\right]dt + n\sigma Y_n(t)d\widetilde{W}(t) \\
&\quad - (n-1)\left(r+\frac{1}{2}\sigma^2 n\right)Y_n(t)dt \\
&= n\sigma Y_n(t)d\widetilde{W}(t)
\end{aligned}$$

(d) Determine, using part (c), the Radon-Nikodym process $Z_t = \left.\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right|_t$

2. Continued

Commentary on Question:
Candidates performed as expected. Most candidates were not able to derive the Z_t *for the specific scenario. Some provided the general formula of the Radon-Nikodym process. One common mistake was an error in the appropriate sign of the first integral.*

$$\gamma(t) = \frac{\left((n-1)\left(0+\frac{1}{2}1^2 n\right)\right)}{1*n} = \frac{n-1}{2}$$

$$\begin{aligned} Z_t &= e^{-\int_0^t \gamma(u)dW(u) - \frac{1}{2}\int_0^t \gamma^2(u)du} \\ &= e^{-\frac{n-1}{2}W(t) - \frac{1}{2}\left(\frac{n-1}{2}\right)^2 t} \\ &= e^{-\frac{n-1}{2}W(t) - \frac{(n-1)^2}{8}t} \end{aligned}$$

(e) Determine a closed-form expression for today's price of $V(T)$.

Commentary on Question:
Candidates performed below expectation on this part. Many candidates did not attempt or stopped after writing a few lines of work. A key insight was to recognize that exponential within V(T) *was the form of the Radon-Nikodym process from part (d), when* n = 8.

Since $r = 0$, the expression for the current price simplifies to:

$$\begin{aligned} V(0) &= E^{\mathbb{Q}}[V(T)] \\ &= E^{\mathbb{Q}}\left[e^{-\frac{49}{8}T - \frac{7}{2}W(T)} S^8(T)\right] \end{aligned}$$

Notice from part (d) that when $n = 8$, $Z_t = e^{-\frac{8-1}{2}W(t) - \frac{(8-1)^2}{8}t} = e^{-\frac{7}{2}W(t) - \frac{49}{8}t}$

So we can solve for the appropriate price through a change of numeraire and find,

$$V(0) = E^{\mathbb{Q}}[Z_T S^8(T)] = E^{\mathbb{Q}^A}[S^8(T)]$$

From part (c), however, we determined that $S^8(T)$ is a martingale under $\mathbb{Q}^A$.

Therefore, $V(0) = S^8(0)$

3. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

Sources:

Chin pg 1-3, 73, 111, 115. Neftci 170

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a)

(i) Write down the 3 conditions satisfied by a sigma-algebra.

(ii) Explain how a filtration is defined in terms of an indexed collection of sigma-algebras.

(iii) Determine whether $\mathcal{F}_t, 0 \le t \le 1$, is a filtration on Ω.

Commentary on Question:

Overall performance on this question was below average, with some candidates not attempting or only partially attempting this question. Of those who attempted, (i) was generally answered partially correctly, with most candidates getting 1 to 2 of the 3 conditions correct, (ii) was generally answered correctly and for (iii), most candidates were unable to provide the counterexamples to prove that $\mathcal{F}_t$ was not a filtration on Ω.

(i)

We check the following conditions (Definitions 1.2 and 1.4):

(a) $\emptyset \in \mathcal{F}_t$ for all $0 \le t \le 1$
(b) The complement of any set in $\mathcal{F}_t$ is also in $\mathcal{F}_t$ for all $0 \le t \le 1$
(c) <u>Countable</u> unions of sets in $\mathcal{F}_t$ are also in $\mathcal{F}_t$ for all $0 \le t \le 1$

(ii)

$\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $0 \le s \le t \le 1$

3. Continued

(iii)
Conditions (b) and (c) are not satisfied, since e.g. $[0, s] \cup [t, 1]$ and (s, t) aren't in $\mathcal{F}_t$.

(b) Show that $X_t = \frac{1}{3} W_t{}^3 - \int_0^t W_s \, ds$.

Commentary on Question:
Overall performance on this question was very strong. Most candidates that attempted the question was able to correctly use Ito's lemma to prove the result. Alternative solutions were accepted as long as the final result was derived correctly.

Solution:

We provide a solution that uses Ito Lemma and note that an alternative approach would be to use the definition of the Ito integral and first principles.

Let $f(W_t) = \frac{1}{3} W_t{}^3$, then

$$df = \frac{\partial f}{\partial W_t} dW_t + \frac{1}{2} \frac{\partial^2 f}{\partial W_t{}^2} dt = W_t{}^2 dW_t + W_t dt$$

therefore

$$X_t = \frac{1}{3} W_t{}^3 - \int_0^t W_s \, ds.$$

(c) Compute the first two (raw) moments of $\frac{1}{3} W_t{}^3$.

Commentary on Question:
Candidates performed moderately on this part. Most were able to determine the first raw moment. Few were able to correctly solve for the second raw moment. Many candidates attempted to solve the second moment, but was unable to get to the finished solution. Reasonable alternative approaches that resulted in the correct conclusion were also awarded points.

We know that

$$E\left(\frac{1}{3} W_t{}^3\right) = 0$$

from the fact that $W_t \sim N(0, t)$ for any $t > 0$.

To compute $E\left(\frac{1}{9} W_t{}^6\right) = \frac{1}{9} E\left(W_t{}^6\right)$, use the mgf for a normal variable with $\sigma^2 = t$:

3. Continued

$$m(\theta) = E\left(e^{W_t\theta}\right) = e^{\frac{t\theta^2}{2}} = 1 + \left(\frac{t\theta^2}{2}\right) + \frac{1}{2!}\left(\frac{t\theta^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{t\theta^2}{2}\right)^3 + \cdots$$

but also note that

$$m(\theta) = \sum_{x=0}^{\infty} \frac{E({W_t}^x)}{x!}\theta^x$$

from which it follows that

$$E\left(\frac{1}{9}{W_t}^6\right) = \frac{6!}{9}\frac{t^3}{48} = \frac{5t^3}{3}.$$

(d) Show that $Y_t = \int_0^t W_s\, ds$ does not have independent increments.

Commentary on Question:
Candidates found this question very challenging with most obtaining little to no points. A minority of candidates were given partial points for setting up the expectation of independent increments to show dependence. However, most candidates were unable to proceed further to solve the expectation equation. Reasonable alternative approaches that resulted in the correct conclusion were awarded points.

To show that Y_t and $Y_T - Y_t$ are not independent, where $0 < t < T$, compute

$$\begin{aligned} E\big(Y_t(Y_T - Y_t)\big) &= E\left(\int_0^t W_s\, ds \times \int_t^T W_s\, ds\right) = E\left(\int_0^t W_s\, ds \times \int_0^{T-t} W_{s+t}\, ds\right) \\ &= E\left(\int_0^t \int_0^{T-t} W_s W_{u+t}\, du\, ds\right) = \int_0^t \int_0^{T-t} E(W_s W_{u+t})\, du\, ds \\ &= \int_0^t \int_0^{T-t} E(W_s(W_{u+t} - W_s))\, du\, ds + \int_0^t \int_0^{T-t} E\left({W_s}^2\right) du\, ds \\ &= \int_0^t \int_0^{T-t} s\, du\, ds = \frac{(T-t)t^2}{2} \neq 0. \end{aligned}$$

4. Learning Objectives:

2. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(2a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

(2f) Understand and be able to apply various model calibration techniques under both risk-neutral and real-world measures

Sources:

Fixed Income Securities: Valuation, Risk , and Risk Management, Veronesi, Pietro, 2010
Understanding the Connection between Real-World and Risk-Neutral Scenario Generators

Commentary on Question:

The question assesses candidates' comprehension of various interest rate models and measures for pricing interest rate derivatives.

Solution:

(a) Recommend whether it would be more appropriate to use a real-world model or a risk-neutral model for the following purposes.

(i) Measuring a 5% worst-case interest rate level for funding status

(ii) Valuation of derivatives

Commentary on Question:

The majority of candidates identified the correct model, but only a few provided reasons.

(i) Measuring a 5% worst-case interest rate level for funding status, and Real-world generators were created to provide realistic stochastic simulation of the path of future economic conditions through time. Each path is a scenario, and such realistic but stochastically generated scenarios can be used in financial models to quantify risk. By running simulations using such scenarios, one can try to answer the questions “How bad could things get” and “How likely is that?” in the context of risk management for a financial institution. **Therefore, real-world scenario generators are fit for the purpose of simulation across time.**

4. Continued

(ii) Valuation of derivatives.
Risk-neutral generators were created for market-consistent valuation of options and other derivatives with an uncertain payoff. A market-consistent valuation method is one which reproduces the market price of market-traded instruments including options and derivatives. Not all such options and derivatives are market-traded and have observable market prices. When buying or selling a derivative that is not traded widely, it is very useful to determine its value by comparison with similar derivatives that are widely traded. It is useful to determine a "market-consistent" price or value. Risk-neutral methods, and generators based upon them, allow determination of a market-consistent value without making any assumption about the market price of risk. That is useful because the price of risk is not directly observable, but the market prices used to calibrate a risk-neutral generator are observable. **Therefore, risk-neutral scenario generators are fit for the purpose of valuing an item with an uncertain payoff in a market-consistent way.**

(b) Determine the value of r_{i+1} that would produce the 5th percentile of the distribution of the funding ratio.

Commentary on Question:
Most candidates failed to mention that a decrease in rates would cause the funding ratio to deteriorate. Consequently, some candidates did not correctly identify the critical value for a 5% VaR. Only very few candidates correctly calculated the interest rate level.

The following equations are derived from real world generators identified in part (b) of the real world.

$$dr_t = \gamma(\bar{r} - r_t)dt + \sigma dX_t$$
$$r_{t+1} = \gamma\bar{r} + (1-\gamma)r_t + \epsilon_{t+1}, \text{where } \epsilon_{t+1} \sim N(0, \sigma^2)$$

The funding ratio is calculated by dividing assets by liabilities (A/L). Because liabilities are more affected by changes in interest rates (with assets being 102% * 37% < liabilities at 100%), a decrease in rates will cause the funding ratio to worsen because liabilities will increase more than assets. As a result, we will be utilizing a 5% VaR.

4. Continued

$$r_1 = \gamma\bar{r} + (1-\gamma)r_0 + \epsilon_1 = 0.3261 \times 0.0509 + (1-0.3261) \times 0.0302 + \epsilon_1$$
$$= 0.03695 + \epsilon_1$$
$$VaR_{5\%}(\epsilon_1) = N^{-1}(0.05, 0, 0.021) = -0.03635$$
$$\therefore r_1^{Worst\ 5\%} = 0.03695 - 0.03635 = 0.06\%$$

(c)

(i) Verify that the calibrated CIR model does not allow negative rates.

(ii) Plot a chart displaying the zero-coupon bond yield for both the Vasicek and CIR models for each maturity from 1-year to 10-years with an increment of 1 year.

The investment group is considering bond options that have a maturity of 1 year and an underlying bond with a maturity of 10 years.

(iii) Calculate the values of the call option and put option with strike price 64.06 under the Vasicek interest rate model.

Commentary on Question:
A partial score was awarded to many candidates who correctly calculated a portion of the solution.

i) $\gamma^* \cdot \bar{r} = 0.25 \times 6.59\% = 0.016475 > 0.0067 = \frac{\alpha}{2}$

ii) ***Vasicek Model***

$$Z(r,t;T) = e^{A(t,T)-B(t,T)\times r}$$
$$B(t;T) = \frac{1-e^{-\gamma^*(T-t)}}{\gamma^*},$$
$$A(t;T) = \left(B(t;T) - (T-t)\right)\left(\bar{r}^* - \frac{\sigma^2}{2\gamma^{*2}}\right) - \frac{\sigma^2 \cdot B(t,T)^2}{4\gamma^*}$$

CIR Model

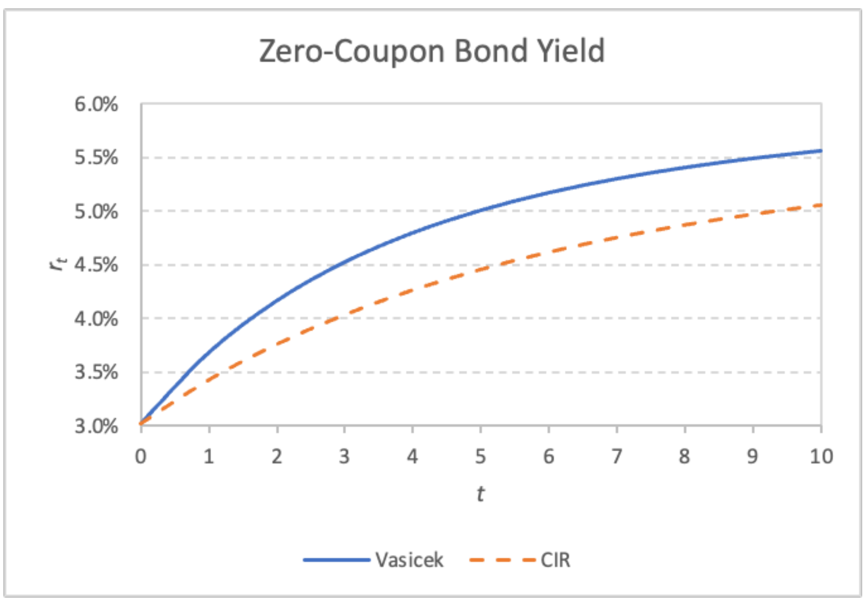
$$Z(r,t;T) = e^{A(t,T)-B(t,T)\times r}$$
$$B(t;T) = \frac{2\left(e^{\psi_1(T-t)}-1\right)}{(\gamma^*+\psi_1)(e^{\psi_1(T-t)}-1)+2\psi_1},$$
$$A(t;T) = 2\frac{\bar{r}^*\gamma^*}{\alpha} log\left(\frac{2\psi_1 e^{(\psi_1+\gamma^*)\frac{(T-t)}{2}}}{(\gamma^*+\psi_1)(e^{\psi_1(T-t)}-1)+2\psi_1}\right)$$
$$\psi_1 = \sqrt{(\gamma^*)^2 + 2\alpha}$$

4. Continued

Using

$$yield = -\frac{\ln\big(Z(r_0, 0; T)\big)}{T}$$

Time	Yield (Vasicek)	Yield (CIR)
0	3.02%	3.02%
1	3.68%	3.43%
2	4.16%	3.76%
3	4.52%	4.04%
4	4.79%	4.27%
5	5.00%	4.46%
6	5.17%	4.62%
7	5.30%	4.76%
8	5.40%	4.87%
9	5.49%	4.97%
10	5.56%	5.06%



iii) Risk Neutral parameters will be utilized as identified in part (b).

$$V^{Call}(r_0, 0) = Z(r_0, 0; T_B)N(d_1) - KZ(r_0, 0; T_O)N(d_2)$$
$$V^{Put}(r_0, 0) = KZ(r_0, 0; T_O)N(-d_2) - Z(r_0, 0; T_B)N(-d_1)$$
$$d_1 = \frac{1}{S_Z(T_O)}\log\left(\frac{Z(r_0, 0; T_B)}{KZ(r_0, 0; T_O)}\right) + \frac{S_Z(T_O)}{2}, d_2 = d_1 - S_Z(T_O)$$

4. **Continued**

$$S_Z(T_O) = B(T_O; T_B) \times \sqrt{\frac{\sigma^2}{2\gamma^*}\left(1 - e^{-2\gamma^* T_O}\right)}$$

	Value
T_O	1
T_B	10
$A(t,T)$	-0.4298
$B(t,T)$	2.1165
$Z(0,T)$	0.5737
$Z(t,T)$	0.6103
$S_Z(T_O)$	0.0377
d_1	-1.9287
d_2	-1.9664
$N(d_1)$	0.0269
$N(d_2)$	0.0246
$N(-d_1)$	0.9731
$N(-d_2)$	0.9754
Call	0.0219
Put	4.3971

5. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

2. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(2f) Understand and be able to apply various model calibration techniques under both risk-neutral and real-world measures

Sources:

An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014 (Ch. 9, Ch10)

Understanding the Connection between Real-World and Risk-Neutral Scenario Generators

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010

Commentary on Question:

This question tests candidates' understanding of the fundamentals of stochastic differential and model calibration under both risk-neutral and real-world measures. Most candidates performed very well in part (a) and (b), but not many candidates earned points in part (c) and (d).

Solution:

(a) Show that

(i) $E(r_t|F_s) = r_s e^{-\gamma(t-s)} + \bar{r}(1 - e^{-\gamma(t-s)})$

(ii) $Var(r_t|F_s) = \frac{\sigma^2}{2\gamma}\left[1 - e^{-2\gamma(t-s)}\right]$

Commentary on Question:

Most Candidates successfully derived the formulas of the expectation and variance.

5. **Continued**

$$dr_u = \gamma(\bar{r} - r_u)du + \sigma dX_u$$

$$e^{-\gamma(t-u)}dr_u = e^{-\gamma(t-u)}\gamma(\bar{r} - r_u)du + e^{-\gamma(t-u)}\sigma dX_u$$

$$e^{-\gamma(t-u)}dr_u + e^{-\gamma(t-u)}\gamma r_u du = e^{-\gamma(t-u)}\gamma(\bar{r})du + e^{-\gamma(t-u)}\sigma dX_u$$

$$d\left(e^{-\gamma(t-u)}r_u\right) = e^{-\gamma(t-u)}\gamma(\bar{r})du + e^{-\gamma(t-u)}\sigma dX_u$$

Integration on both sides of the equation from s to t

$$r_t = r_s e^{-\gamma(t-s)} + \bar{r}\left(1 - e^{-\gamma(t-s)}\right) + \sigma \int_s^t e^{-\gamma(t-u)}\, dX_u$$
$$E(r_t|F_s) = r_s e^{-\gamma(t-s)} + \bar{r}(1 - e^{-\gamma(t-s)})$$
$$as\ E\left\{\sigma \int_s^t e^{-\gamma(t-u)}\, dX_{\mathrm{u}}\right\} = 0$$

$$Var(r_t|F_s) = E\{[(r_t - E(r_t)^2|F_s\}$$
$$=\sigma^2 \left[\int_s^t e^{-\gamma(t-u)}\, dX_{\mathrm{u}}\right]^2$$

By Ito Isometry,

$$=\sigma^2 \int_s^t e^{-2\gamma(t-u)}\, du$$

$$= \frac{\sigma^2}{2\gamma}\left[1 - e^{-2\gamma(t-s)}\right]$$

(b) Determine the market price of interest risk.

Commentary on Question:
Most candidates successfully identified the formula of market price.

$\lambda(r,t) = \frac{1}{\sigma}(\gamma(\bar{r} - r) - \gamma^*(\overline{r^*} - r))$ = -0.004/0.01=-0.4

5. Continued

(c) Compute the drift and the diffusion of $\frac{dZ}{Z}$ for the risk-neutral process.

Commentary on Question:
Less than half of the candidates successfully identified the diffusion term in the risk-neutral process and calculated correctly. Candidates earned partial credits if they can identify the correct formula or claim the correct drift term.

The drift of $\frac{dZ}{Z}$ or the instantaneous return of the bond, in the risk-neutral world is 4%.

$$\frac{dZ(t,T)}{Z(t,T)} = r_t dt + \sigma_Z(t,T) dX_t$$

$$\sigma_Z(t,T) = -B(t;T)\sigma$$

The diffusion of $\frac{dZ}{Z}$ in the risk-neutral world
$= -B(0;10)\sigma$
$= -\frac{1-e^{-0.1\,x\,10}}{0.1}$ x 0.1
=-0.063212

(d) Compute the drift and the diffusion of $\frac{dZ}{Z}$ for the real-world process.

Commentary on Question:
Few candidates preformed perfectly in this part by using the correct equation to move from risk-neutral process to real-world process. Candidates earned partial credits if they can state the drift term didn't change from part (c).

From part (d)

$$\frac{dZ}{Z} = 0.04dt - 0.063212Z\ dX$$

When we move to the real world, the return increases by the product of the market price of dZ risk and -0.063212.

The bond price process becomes:
$$\frac{dZ}{Z} = [0.04 + (-0.4\ x - 0.063212)]dt - 0.063212\ dX$$

5. Continued

$$\frac{dZ}{Z} = 0.065285\ dt - 0.063212\ dX$$

The drift increases from 4% to 6.5285% as we move from the risk-neutral world to the real world

The diffusion in the real world
= $-B(0;10)\sigma$
= $-\frac{1-e^{-0.1\ x\ 10}}{0.1}$ x 0.1
=-0.063212

6. Learning Objectives:

2. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(2b) Understand and be able to apply various one-factor interest rate models and various simulation techniques including Euler-Maruyama discretization and transition density methods

(2e) Understand model selection and the appropriateness to the specific purpose

(2f) Understand and be able to apply various model calibration techniques under both risk-neutral and real-world measures

Sources:

Interest rate model calibration,
Veronesi

Commentary on Question:

Overall, candidates either left the question blank or attempted only part of it. For those that attempted this question, candidates performed better where there were formulas involved. However, when it came to the qualitative aspects of the question, there were fewer candidates that were able to provide appropriate explanations.

Solution:

(a) Describe the assumptions made in the chosen real-world parameter estimation method.

Commentary on Question:

Full marks were given if each of the following assumptions were provided. Many candidates recognized the need for using the maximum likelihood function and regression. Many candidates failed to identify that a major assumption is that the rates follow a normal distribution.

6. **Continued**

Assumptions:

- Daily yields of 3 months anualized rates follow normal distribution.
  - $r_{t+s}|r_t$ is normally distributed with mean $\bar{r} + (r_t - \bar{r})exp\ (-\gamma s)$ and variance $\frac{\sigma^2}{2\gamma}(1 - exp(-2\gamma s))$
  - The conditional pdf of $r_{i\Delta}|r_{(i-)\Delta}, i = 1,2, \ldots$ is normal.
  - We can write the likelihood function of the sample.
  - Minimizing likelihood function is equivalent to regressing
    $$y = (r_\Delta, r_{2\Delta}, \ldots, r_{n\Delta})^T$$
    on $x = \left(r_0, r_\Delta, \ldots, r_{(n-1)\Delta}\right)^T$.
  - Above is true if the contribution of r_0 to the likelihood function is small or in another word sample is very large.

(b) Estimate the parameters of your model.

Commentary on Question:
In order to receive full marks, the candidate needed to identify the formulas below and accurately use them to obtain the correct results. Full credit was given for candidates that used Euler's method to approximate gamma.

Estimated regression parameters and Vasicek model parameters are related as follows

$$\gamma = -\frac{ln\left(\hat{\beta}^*\right)}{\Delta}$$
$$\bar{r} = \frac{\hat{\alpha}^*}{1 - \hat{\beta}^*}$$
$$\sigma = \sqrt{\frac{2\gamma{\hat{\sigma}^*}^2}{1 - {\hat{\beta}^*}^2}}$$

From the given R output

$$\hat{\alpha}^* = 0.0001815, \hat{\beta}^* = 0.9964045, \hat{\sigma}^* = 0.001300116$$

Plugging these values in the formulas we obtain

$$\gamma = 0.90769249, \bar{r} = 0.05049257, \sigma = 0.02067589$$

6. Continued

(c) Describe the procedure employed in risk-neutral model calibration.

Commentary on Question:
Many candidates identified at least one aspect of the procedure below. Partial credit was given in most cases if the candidate could identify the need for non-linear regression, least-squares regression, and minimizing the difference between modeled rates vs. market rates. Few candidates identified the initial formula below or the sensitivity of the initial guess.

In the method Vasicek yield rates are calculated using the formula

$$r(t) = \frac{\left(r(0)B(0;t) - A(0;t)\right)}{t}$$

In the A(0;t) and B(0;t) are as given in the formula.
Then use the non-linear least square regression method to minimize the distance between obsereved values of r(t) with the expected values of r(t) with respect to parameters.

In studies this method performs better than fitting observed bond prices to its theoretical prices under Vasicek model.

Also, the non-linear leastsquare estimation method is quite sensitive to initial guesses, many different initial guesses

(d) Estimate the parameters of your new model.

Commentary on Question:
Full marks were given if the formula below was identified and used appropriately. Many candidates used the formula directly without commentary for why the formula was appropriate.

The output does not contain estimated model parameters but it contains standard error and t-value; multiplying these items together we obtain estimates.

$$\gamma^* = 0.031458 * 15.49 = 0.487311$$
$$\bar{r}^* = 0.001385 * 50.90 = 0.070482$$

6. Continued

(e) Determine whether the fitted models are adequate.

Commentary on Question:
Full credit was given if the candidate could appropriately identify that both the real-world and risk-neutral world models are appropriate. Many candidates commented on the p-values, however not very many candidates commented on R-squared.

From the diagnostic statistics for realworld estimate we see that p-value for the test H_0: $\beta^* = 0$ is almost zero so that test is rejected with certainty.

Also R-squared is close to 1 so model describes the data almost perfectly.

For the risk neutral parameter estimation both p-values are close to zero so model is perfect.

7. **Learning Objectives:**

3. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(3a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(3c) Demonstrate an understating of the different approaches to hedging – static and dynamic.

(3f) Appreciate how hedge strategies may go awry.

(3h) Compare and contrast the various kinds of volatility, e.gl, actual, realized, implied and forward, etc.

Sources:

Derman and Miller, Ch. 2-3

Marroni and Perdomo, Ch. 7

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Determine the cost to replicate the strategy today with a static hedge.

The 1 year point-to-point strategy can be replicated by buying the at-the-money call and shorting the out-of-the-money call. The put prices are irrelevant.

Initial Hedge Cost = 374 – 214 = 160

(b) Calculate A and B at time t = 0.5.

Commentary on Question:

Candidates performed poorly on this question part. While many could identify the correct stochastic process for the P&L, few performed all of the necessary steps to receive full credit.

7. **Continued**

When hedging the option using realized volatility, the future P&L is governed by:

$$dP\&L = dV_I - \Delta_R\, dS - \Delta_R SD\, dt - (V_I - \Delta_R S) r\, dt$$

As the index is pays no dividends, this process reduces to:

$$dP\&L = dV_I - \Delta_R\, dS - (V_I - \Delta_R S) r\, dt$$

Hence, $A = -\Delta_R$ and $B = -(V_I - \Delta_R S)r$.

Δ_R is simply the delta of the at-the-money call minus the delta of the out-of-the-money call, or:

$$\Delta_R = e^{-D\tau} N\left(d_{1,ATM}\right) - e^{-D\tau} N\left(d_{1,OTM}\right) = N\left(d_{1,ATM}\right) - N(d_{1,OTM})$$

where

$$d_{1,ATM} = \frac{ln\left(\frac{S}{K}\right) + \left(r - D + \frac{1}{2}\sigma_R^2\right)\tau}{\sigma_R\sqrt{\tau}} = \frac{ln\left(\frac{4500}{4200}\right) + \left(0.02 + \frac{1}{2}(0.25)^2\right)(0.5)}{(0.25)\sqrt{0.5}}$$
$$= \frac{0.0946}{0.1768} = 0.54$$

and $N\left(d_{1,ATM}\right) = 0.7054$ from the Z table. Similarly, $N\left(d_{1,OTM}\right) = N(0.02) = 0.5080$ and we have $\Delta_R = 0.7054 - 0.5080 = 0.1974$

V_I is the value of the option at time t= 0.5 using implied volatilties. This is simply the market price of the option at that time, or

$$V_I = Call(K = 4200, \tau = 0.5) - Call(K = 4600, \tau = 0.5) = 452 - 229 = 223$$

Lastly, substituting all relevant values in for the formulas for A and B produces

$$A = -\Delta_R = -0.1974$$

and

$$B = -(V_I - \Delta_R S)r = -\left(233 - (0.1974)(4500)\right)(0.02) = 13.11$$

8. **Learning Objectives:**

3. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(3a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(3e) Analyze the Greeks of common option strategies.

Sources:

QFIQ-120-19: Chapters 6 and 7 of Pricing and Hedging Financial Derivatives, Marroni, Leonardo and Perdomo, Irene, 2014

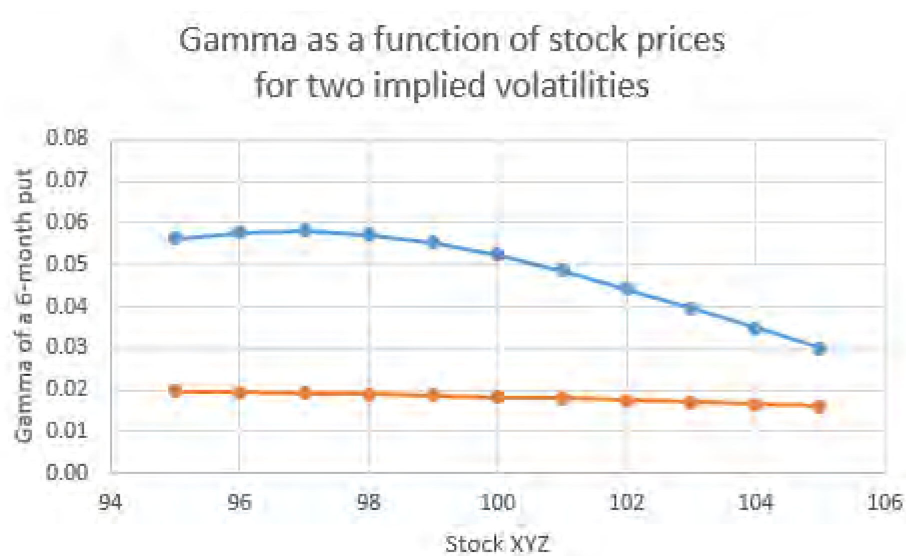
Commentary on Question:

This question tests candidates' knowledge on the Greeks of common equity options. Since the Greeks are a function of many variables (such as the stock price, strike price, interest rate, implied volatility and time-to-maturity), this question effectively tests candidates' ability to analyze the sensitivity of the Greeks with respect to one underlying variable at a time.

Solution:

(a) Consider a put option on Stock XYZ, a non-dividend-paying stock. Also assume that:

- The spot price of Stock XYZ is 100 and the option strike is 100.
- The continuously compounded risk-free interest rate is 5%.



8. Continued

The chart above shows the option Gamma as a function over a range of stock prices under two different implied volatilities. For part (a), also assume that:

- The time-to-maturity is six months.
- One curve is the option Gamma under 10% implied volatility while the other one is the option Gamma under 30% implied volatility.

Determine which curve corresponds to which implied volatility.

Commentary on Question:
Candidates performed poorly on this part. There are at least three ways to answer this part: (1) Qualitative method through one's understanding of Gamma. (2) Quantitative method by computing Gamma at one of the points on the curve and then infer which curve is which; (3) Analytical method by deriving the partial derivative of Gamma with respect to the volatility. Full or partial credits were awarded to candidates depending upon the method they used and the completeness of their solution.

1 Qualitative method:
Based on the range prescribed, the graph asks for the Gamma curve for an at-the-money put option. The lower the volatility, the more likely the underlying stock is to stay around at the option strike, and thus the less likely the option is to flip from in-the-money to out-of-the-money and vice versa. Thus, in a lower volatility regime, the Gamma will be higher because a change in stock price is more likely to trigger bigger change in the Delta. Therefore, the line above (the blue line with triangles) is the Gamma curve under the 10% implied volatility and the line below (the orange line with squares) is the Gamma curve under the 30% implied volatility.

2 Quantitative method:

$$\text{Gamma} = \frac{N'(d_1)}{S\sigma\sqrt{T-t}}$$

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

Using σ = 10%, r = 5%, T – t = 0.5 and S = K = 100
We have Gamma = 0.0523, which corresponds to the gamma value of blue line at S = 100.
Therefore, the blue line is for Gamma under 10% implied vol and the orange line is for Gamma under 30% vol.

8. **Continued**

<u>3 Analytical method:</u>

$$Gamma = \frac{1}{S\sqrt{2\pi(T-t)}} * \frac{e^{\frac{-d_1^2}{2}}}{\sigma}$$

$$d_1 = \frac{\sigma\sqrt{T-t}}{2} + \frac{\ln\left(\frac{S}{K}\right) + r(T-t)}{\sigma\sqrt{T-t}}$$

$$\frac{\partial Gamma}{\partial \sigma} = \frac{1}{S\sqrt{2\pi(T-t)}} * \left(-\frac{e^{\frac{-d_1^2}{2}}}{\sigma^2} - \frac{e^{\frac{-d_1^2}{2}}}{\sigma} * d_1 * \frac{\partial d_1}{\partial \sigma}\right)$$

$$\frac{\partial Gamma}{\partial \sigma} = \frac{1}{S\sqrt{2\pi(T-t)}} * \left(-\frac{e^{\frac{-d_1^2}{2}}}{\sigma^2} - \frac{e^{\frac{-d_1^2}{2}}}{\sigma} * d_1 * \left(\frac{\sqrt{T-t}}{2} - \frac{\ln\left(\frac{S}{K}\right) + r(T-t)}{\sigma^2\sqrt{T-t}}\right)\right)$$

Because σ appears in d_1, its impact on Gamma depends upon the values of other variables. *There is no monotonic relationship between Gamma and σ.* Therefore, conclusion of "Gamma will be higher with lower σ because σ is in the denominator of Gamma" is incomplete — it could be wrong when values of other variables are such that $\frac{\partial Gamma}{\partial \sigma} > 0$. See note below.

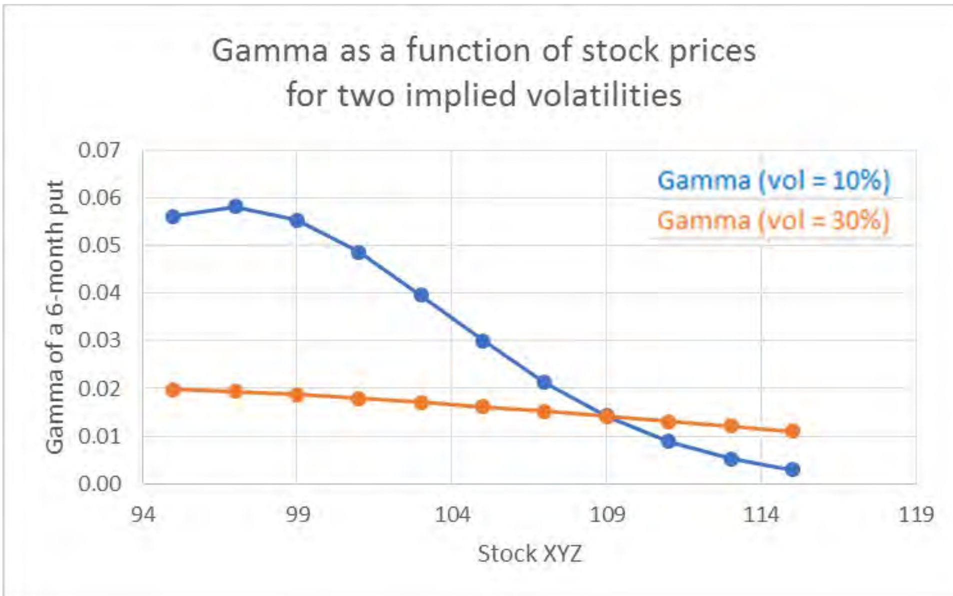
For this question, we can use S = K = 100, T – t = 0.5, r = 5%, $\sigma = 10\%$ to get

$$\frac{\partial Gamma}{\partial \sigma} = -0.43$$

This helps us to conclude that when stock price is close to the strike price, Gamma will decrease when implied vol increases. So the blue curve is under 10% implied vol and the orange curve is under 30% implied vol.

Note: Though this analytical approach is more complicated than the other two approaches, the purpose here is to show that one should bear in mind d_1 and d_2 when evaluating the impact on the greeks with respect to changes of one variable because d_1 and d_2 are functions of all variables that impact the option price and Greeks. For example, if S = 110 and the values of all other variables are the same as before, we will have $\frac{\partial Gamma}{\partial \sigma} = 0.35$, implying that Gamma will increase when implied vol increases, as confirmed by the chart below:

8. Continued



(b)

8. Continued

The chart above shows the option Theta as a function over a range of stock prices under two different implied volatilities. For part (b) and part (c), assume that:

- The spot price of Stock XYZ is 100 and the put option strike is 100.
- The time-to-maturity is one year.
- One curve is the option Theta with a 20% implied volatility while the other curve is the option Theta with a 30% implied volatility.

Determine which curve corresponds to which implied volatility.

Commentary on Question:
Candidates performed poorly on this part, which tests candidates' ability to assess how Theta will change with respect to changes in the implied vol. Because this part is similar to Part (a), we illustrate three solutions. Full or partial credits were awarded to candidates depending upon the method they used and the completeness of their solution.

1 Qualitative Solution:
Based on the prescribed range, the graph asks for all three cases: in-the-money, at-the-money, and out-of-the-money.

For out-of-the-money and in-the-money cases, when the volatility is low, the probability of the underlying stock crosses below or above the strike is also low. We thus expect to see low sensitivity of option value to time to expiry. For at-the-money, the sensitivity is higher.

If volatility starts to move higher, we should expect to see a higher probability of option expiring in-the-money and expiring out-of-the-money, and thus higher sensitivity to time to expiry.

Therefore, the line above (blue line with triangles) is the Theta curve with lower implied volatility while the line below (orange line with squares) is the Theta curve with higher implied volatility.

2 Quantitative Solution:
'
The theta of a plain vanilla put is equal to:

$$-\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} + rKe^{-r(T-t)}N(-d_2)$$

8. Continued

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Using σ = 20%, r = 5%, T – t = 1 and S = K = 100
We have Theta = –1.66, which corresponds to the theta value of blue line at S = 100.
Therefore, the blue line is for Theta under 20% implied vol and the orange line is for Theta under 30% vol.

<u>3 Analytical Solution</u>
The equation for $\frac{\partial Theta}{\partial \sigma}$ is much more complicated due to Theta involving both d_1 and d_2 and is omitted here. Again, the purpose here is to show that one cannot draw conclusions of "Theta will be more negative with higher σ due to σ appearing explicitly in the numerator of the first term of the Theta formula". We must also consider the implicit impact of σ through d_1 and d_2. The impact on Theta is through the net effect of these offsetting forces.

(c) Plot the two Theta curves.

Commentary on Question:
Candidates performed poorly on this part. This part is an extension of Part (b) that tests candidates' ability to (re)-draw the Theta curve given in Part (b). Full credit is given for each correctly-drawn curve and zero credit otherwise.

8. Continued

Stock (S)	100
Strike (K)	100
Interest Rate (r)	0.05
Dividend (d)	0.00
Maturity (T)	1.0
Implied Vol1	20%
Implied Vol2	30%

The theta of a plain vanilla put is equal to:

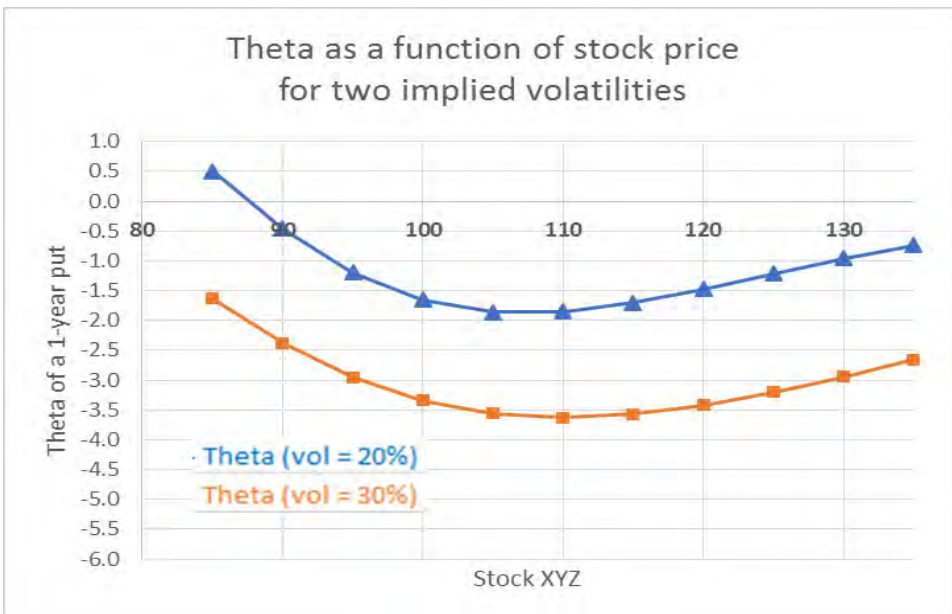
$$-\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} + rKe^{-r(T-t)}N(-d_2)$$

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

Stock	Theta (vol=20%)	Theta (vol=30%)	d1 (vol=20%)	N'(d1) (vol=20%)	d2(vol=20%)	N(-d2) (vol=20%)	d1 (vol=30%)	N'(d1) (vol=30%)	d2(vol=30%)	N(-d2) (vol=30%)
85	0.50	-1.63	-0.4626	0.3585	-0.6626	0.7462	-0.2251	0.3890	-0.5251	0.7002
90	-0.46	-2.38	-0.1768	0.3928	-0.3768	0.6468	-0.0345	0.3987	-0.3345	0.6310
95	-1.19	-2.96	0.0935	0.3972	-0.1065	0.5424	0.1457	0.3947	-0.1543	0.5613
100	-1.66	-3.35	0.3500	0.3752	0.1500	0.4404	0.3167	0.3794	0.0167	0.4934
105	-1.86	-3.56	0.5940	0.3344	0.3940	0.3468	0.4793	0.3557	0.1793	0.4289
110	-1.86	-3.63	0.8266	0.2835	0.6266	0.2655	0.6344	0.3262	0.3344	0.3691
115	-1.71	-3.57	1.0488	0.2302	0.8488	0.1980	0.7825	0.2937	0.4825	0.3147
120	-1.47	-3.42	1.2616	0.1800	1.0616	0.1442	0.9244	0.2602	0.6244	0.2662
125	-1.21	-3.20	1.4657	0.1363	1.2657	0.1028	1.0605	0.2274	0.7605	0.2235
130	-0.96	-2.94	1.6618	0.1003	1.4618	0.0719	1.1912	0.1962	0.8912	0.1864
135	-0.74	-2.66	1.8505	0.0720	1.6505	0.0494	1.3170	0.1676	1.0170	0.1546



(d) For part (d), you want to evaluate the sensitivity of Rho to the interest rate levels because the interest rate could potentially go much higher or much lower. Assume the following:

- The time-to-maturity is one year.
- Consider the Rho under two different interest rate levels at 3% and 10%, respectively.

Plot the two Rho curves as a function of stock prices.

8. Continued

Commentary on Question:
Candidates performed poorly on this part. This part tests candidates' ability to draw the Rho curve with respect to different interest rate. Full credit is given for each correctly-drawn curve and zero credit otherwise.

Stock (S)	100
Strike (K)	100
Dividend (d)	0.0
Time to maturity (T)	1.0
Risk Free Rate 1 (r1)	0.03
Risk Free Rate 2 (r2)	0.10
Implied Vol	0.2

$$\text{Rho(put)} = -K(T-t)e^{-r(T-t)}N(-d_2)$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

Stock	Rho (r1=3%)	Rho (r2=10%)	d2 (r1=3%)	N(-d2) (r1=3%)	d2 (r2=10%)	N(-d2) (r2=10%)
90	-66.31	-49.81	-0.4768	0.6832	-0.1268	0.5505
95	-56.46	-40.08	-0.2065	0.5818	0.1435	0.4429
100	-46.59	-31.18	0.0500	0.4801	0.4000	0.3446
105	-37.30	-23.51	0.2940	0.3844	0.6440	0.2598
110	-29.04	-17.22	0.5266	0.2993	0.8766	0.1904
115	-22.03	-12.30	0.7488	0.2270	1.0988	0.1359
120	-16.32	-8.58	0.9616	0.1681	1.3116	0.0948
125	-11.83	-5.86	1.1657	0.1219	1.5157	0.0648
130	-8.41	-3.93	1.3618	0.0866	1.7118	0.0435
135	-5.87	-2.60	1.5505	0.0605	1.9005	0.0287

8. Continued


Rho as a function of stock price
for two interest rates
0.0
-10.0
-20.0
-30.0
-40.0
-50.0
-60.0
-70.0
90
95
100
105
110
115
120
125
130
135
Rho of 1-year put
Stock XYZ
Put Rho (r = 10%)
Put Rho (r = 3%)

9. **Learning Objectives:**

3. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(3a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(3e) Analyze the Greeks of common option strategies.

(3g) Describe and explain some approaches for relaxing the assumptions used in the Black-Scholes-Merton formula.

Sources:

The Volatility Smile, Derman, Emanuel and Miller, Michael B., 2016

QFIQ-120-19: Chapters 6 and 7 of Pricing and Hedging Financial Derivatives, Marroni, Leonardo and Perdomo, Irene, 2014

QFIQ-115-17: Which Free Lunch Would You Like Today, Sir?: Delta Hedging, Volatility Arbitrage and Optimal Portfolios

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Calculate the Monte Carlo step volatility based on the market implied volatilities.

Commentary on Question:
Most candidates did well on this part

There are multiple ways to calculate the Monte Carlo step volatilities.

For step one, the volatility is the same as the market implied. To compute the step-two volatility, one matches the total variance in the market for 2nd period with the total variance along with a Monte Carlo path from the evaluation date to 2nd period, as shown below -

$$\text{vol\_mkt}^2(2d) * 2d = \text{vol}^2(1) * dt + \text{vol}^2(2) * dt$$

The process becomes iterative, which can be easily done in Excel using formulas.

$$\text{vol}^2 * (t - s) = \text{vol\_mkt}^2(t) * t - \text{vol\_mkt}^2(s) * s$$

Essentially it is a (N-1)-period forward 1-period volatility.

9. **Continued**

The last formula can be implemented in Excel as:

$$Vol^2 = {vol_{mkt}}^2(t) * t - {vol_{mkt}}^2(t-1) * (t-1) \; for \; t = 2,3, \ldots ,6$$

Or

$$Vol^2 = {vol_{mkt}}^2(t) * t - Vol^2(t-1) - Vol^2(t-2) - \cdots - Vol^2(1) \; for \; t = 2,3, \ldots ,6$$

Both solutions got full credit

(b) Determine whether such a stochastic volatility model creates a volatility smile in the Monte Carlo simulation and how.

Commentary on Question:
Candidates did not do well on this part. Problem was not realizing the relationship between volatility and paths ending ATM/OTM

Using a stochastic volatility model can capture a volatility smile. For example,

1. In one Monte Carlo simulation path, volatility drops sharply and stays low for most of the path. Consequently, the stock price is not likely to change materially and will reach the end of the path relatively near its ATM level.
2. In another Monte Carlo simulation path, volatility increases drastically, and stays high for most of the path. Consequently, the stock price is likely to move significantly and reach the end of the path either at a very high or a very low level.

When averaging across all the Monte Carlo simulation paths, paths ending far from the ATM level will have experienced, on average, a higher volatility than paths ending near ATM. Therefore, a volatility smile is generated.

To create the volatility smile, the vol-of-vol stochastic process has been constructed. This can be done by calibrating the stochastic volatility model to the market prices observed in the market.

(c) List two major disadvantages of using a stochastic volatility model for pricing options.

Commentary on Question:
Candidates did not do well on this part.

9. Continued

A Stochastic volatility model has the following drawbacks:

1. European options prices cannot be reproduced perfectly, only approximately. Stochastic volatility models might be appropriate for exotic options but they may not be appropriate for vanilla options.
2. Calibration of such model can be unstable, resulting in jumps in mark-to-market profit and loss.
3. If one calibrates such models using vanilla option prices, they could still give prices for exotics such as barriers that are not in line with prices observed in the market. Conversely, if one calibrates models using prices for exotics, the models might not be able to get near to the price of vanilla options.

Any two from the above would get full credit. Partial credit was given to answers which did not provide explanation of the drawback(s). Partial credit was given for only one disadvantage listed or if the descriptions were not complete

10. Learning Objectives:

4. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

Learning Outcomes:

(4a) Identify and evaluate embedded options in liabilities, specifically indexed annuity and variable annuity guarantee riders (GMAB, GMDB, GMWB and GMIB).

(4b) Demonstrate an understanding of embedded guarantee risk including: market, insurance, policyholder behavior, and basis risk factors.

Sources:
QFIQ-135-22, QFIQ-128-20

Commentary on Question:
Commentary listed underneath question component.

Solution:

(a) Show that the value of the PPN is:

$$Pe^{-rT}[1 + \alpha\Phi(d_2)]$$

where Φ is the normal cumulative distribution function and

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - q - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}.$$

Commentary on Question:

Many candidates did well on this question. Many candidates recognized that the probability of $S_T > K$ *for the PPN is equal to* $N(d_2)$. *Most candidates then were able to derive the correct value of the PPN.*

The value of the PPN is the expected present value of its payoff at time T.

Under the Black-Scholes framework, the risk-neutral probablity of a call option being in-the-money at maturity is:

$$P(S_T > K) = \Phi(d_2)$$

Therefore,

$$\begin{aligned} e^{-rT}E^{\mathbb{Q}}[Payoff] &= e^{-rT}\left[P\left(1 - \Phi(d_2)\right) + (1 + \alpha)P\Phi(d_2)\right] \\ &= Pe^{-rT}[1 + \alpha\Phi(d_2)] \end{aligned}$$

10. Continued

(b)

(i) Show that the fair step-up rate is:

$$\alpha = \frac{e^{rT} - 1}{\Phi(d_2)}$$

(ii) Show that in the limit as $K \longrightarrow 0$, the fair step-up rate approaches the theoretical minimum value of $e^{rT} - 1$.

Commentary on Question:
Many candidates did well on this question. Candidates generally had no issues solving for the value of the fair step-up rate.

(i) The fair cap rate can be found by setting the value of the PPN equal to P and solving for α:

$$P = Pe^{-rT}[1 + \alpha\Phi(d_2)]$$
$$\Rightarrow \alpha = \frac{e^{rT} - 1}{\Phi(d_2)}$$

(ii) In the limit as $K \longrightarrow 0$, $d_2 \longrightarrow \infty$ and $\Phi(d_2) = 1$, thus α will approach the theoretical minimum value of $e^{rT} - 1$.

(c)

(i) Calculate the fair step-up rate on this PPN.

(ii) Explain whether the fair step-up rate would increase, decrease, or not change if an annual expense charge, $m > 0$, is added to this PPN.

Commentary on Question:
Most candidates did well on part (i). Common mistakes included numerical mistakes when plugging values in the fair step-up rate equation. Most candidates did not perform well on part (ii), as they generally failed to recognize that the expense charge needed to be funded by the step-up rate, and thus the fair step-up rate would decrease.

10. Continued

(i) Plug into expression from part (b):

$$\alpha = \frac{e^{rT} - 1}{\Phi(d_2)}$$
$$= \frac{e^{.04*2} - 1}{\Phi\left(\frac{\ln\left(\frac{1000}{1100}\right) + \left(.04 - .01 - \frac{1}{2} *.15^2\right) * 2}{.15\sqrt{2}}\right)}$$
$$= \frac{0.08329}{\Phi(-0.27252)} = 0.2121$$

(ii) Let $m > 0$ be the expense rate charged by the insurer. Then the fair step-up rate will decrease as seen by the new expression for the fair step-up rate:

$$P = Pe^{-rT}e^{mT}[1 + \alpha\Phi(d_2)] \Rightarrow \alpha = \frac{e^{(r-m)T} - 1}{\Phi(d_2)}$$

Alternatively, it can be reasoned that under an expense charge, $m > 0$, the value of the structure to the policyholder must decrease, which means α must now be lower.

(d)

(i) "Given a fixed step-up rate and strike price, as the risk-free rate increases, the value of the PPN will decrease."

(ii) "Given a fixed step-up rate and strike price, as the implied volatility increases, the value of the PPN will decrease."

Commentary on Question:
Most candidates did not perform well on this part. Most candidates did not recognize that the change in value of PPN is dependent on both the risk-free rate / volatility and the money-ness of the cash-or-nothing call.

(i) Disagree. As the risk-free rate increases, the value of the guaranteed principal portion of the note decreases, but the probability of in-the-moneyness on long call options increases (positive rho on a long cash-or-nothing call). Both effects combined can either increase or decrease the value of the entire PPN.

10. Continued

(ii) Disagree. This is only true if $K \leq S_0$. When K is much greater than S_0, higher implied volatility increases the probability of in-the-moneyness (and thus the value of the PPN).

(e) Show that the fair step-up rate for the PPN-GMDB structure is:

$$\frac{\frac{r}{\mu_x + r}\left[e^{(\mu_x+r)T} - 1\right]}{\Phi(d_2)}$$

Commentary on Question:
Most candidates did not perform well on this question. Many candidates did not attempt this part. Also, common mistakes included no setting up the integral expressions for the mortality risk, or algebraic mistakes during the derivation.

The value of the PPN-GMDB structure is:

$$\begin{aligned}
&Pe^{-rT}[1 + \alpha\Phi(d_2)]_T p_x + P\int_0^T e^{-rt}{}_t p_x \mu_{x+t} dt \\
&= Pe^{-rT}[1 + \alpha\Phi(d_2)]e^{-\mu_x T} + P\int_0^T \mu_x e^{-(\mu_x+r)t} dt \\
&= Pe^{-(\mu_x+r)T}[1 + \alpha\Phi(d_2)] + P\frac{\mu_x}{\mu_x + r}\left[1 - e^{-(\mu_x+r)T}\right]
\end{aligned}$$

The fair step-up rate can be found by setting the value of the PPN-GMDB equal to *P* and solving for α:

$$\begin{aligned}
\alpha &= \frac{e^{(\mu_x+r)T} - \frac{\mu_x}{\mu_x + r}\left[e^{(\mu_x+r)T} - 1\right] - 1}{\Phi(d_2)} \\
&= \frac{\frac{r}{\mu_x + r}\left[e^{(\mu_x+r)T} - 1\right]}{\Phi(d_2)}
\end{aligned}$$

11. **Learning Objectives:**

4. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

Learning Outcomes:

(4a) Identify and evaluate embedded options in liabilities, specifically indexed annuity and variable annuity guarantee riders (GMAB, GMDB, GMWB and GMIB).

(4b) Demonstrate an understanding of embedded guarantee risk including: market, insurance, policyholder behavior, and basis risk factors.

Sources:
QFIC 135-22

Commentary on Question:
This question tests the candidate's understanding of spVA products with a cap and a buffer and floor product design. It gets into how replicating option portfolios can be constructed for such products to support the payoff.

Candidates' performance on this question was not very good.

Solution:

(a) Show that a portfolio of a risk-free bond (with maturity value of S_0) and the following options provides the maturity payoff of this new spVA.

(i) Long P^u units of call option with strike price S_0 (i.e., at-the-money ATM)

(ii) Short P^u units of call option with strike price $S_0(1+\frac{C}{P^u})$ (i.e., out-of-money OTM)

(iii) Short a put option with the strike price $S_0(1-B)$ (i.e., OTM)

Commentary on Question:
Many candidates received partial credit for (a) for specifying the payoff of the various options as they didn't show the work to combine the payoffs for the overall payoff.

11. Continued

The bond maturity payoff is S_0. So, deducting the spVA payoff by S_0 gives the following maturity payoff equations for the portfolio of options

$$Payoff_t^{spVA} - S_0 = \begin{cases} S_0 C & , \quad S_t \geq S_0\left(1+\frac{C}{P^u}\right) \\ P^u(S_t - S_0) & , \quad S_0 \leq S_t < S_0\left(1+\frac{C}{P^u}\right) \\ 0 & , \quad S_0(1-B) \leq S_t < S_0 \\ (S_t - S_0) + S_0 B & , \quad S_t \leq S_0(1-B) \end{cases}$$

A) short P^u units of call option with strike price $S_0(1+\frac{C}{P^u})$ provides the following payoff

$$Payoff_t^{OTM\ call} = \begin{cases} P^u\left(S_0\left(1+\frac{C}{P^u}\right) - S_t\right) & , \quad S_t \geq S_0\left(1+\frac{C}{P^u}\right) \\ 0 & , \quad S_t < S_0\left(1+\frac{C}{P^u}\right) \end{cases}$$

B) long P^u units of ATM call option with strike price S_0 provides the following payoff

$$Payoff_t^{ATM\ call} = \begin{cases} P^u(S_t - S_0) & , \quad S_t \geq S_0 \\ 0 & , \quad S_t < S_0 \end{cases}$$

C) short an OTM put option with strike price $S_0(1-B)$ provides the following payoff

$$Payoff_t^{OTM\ put} = \begin{cases} 0 & , \quad S_t \geq S_0(1-B) \\ (S_t - S_0(1-B)) & , \quad S_t < S_0(1-B) \end{cases}$$

Putting A) and B) together gives the following payoff

$$Payoff_t^{OTM\ call} + Payoff_t^{ATM\ call} = \begin{cases} P^u\left(S_0\left(1+\frac{C}{P^u}\right) - S_t\right) + P^u(S_t - S_0), & S_t \geq S_0\left(1+\frac{C}{P^u}\right) \\ P^u(S_t - S_0) & , \quad S_0 \leq S_t < S_0\left(1+\frac{C}{P^u}\right) \\ 0 & , \quad S_t < S_0 \end{cases}$$

Note that $P^u\left(S_0\left(1+\frac{C}{P^u}\right) - S_t\right) = S_0 C + (S_0 - S_t)P^u$

Thus,

$$Payoff_t^{OTM\ call} + Payoff_t^{ATM\ call} = \begin{cases} S_0 C & , \quad S_t \geq S_0\left(1+\frac{C}{P^u}\right) \\ P^u(S_t - S_0) & , \quad S_0 \leq S_t < S_0\left(1+\frac{C}{P^u}\right) \\ 0 & , \quad S_t < S_0 \end{cases}$$

Lastly, putting A), B) and C) together gives the following payoff

11. Continued

$$Payoff_t^{OTM\ call} + Payoff_t^{ATM\ call} + Payoff_t^{OTM\ put} = \begin{cases} S_0 C & , \quad S_t \geq S_0\left(1+\frac{C}{P^u}\right) \\ P^u(S_t - S_0) & , \quad S_0 \leq S_t < S_0\left(1+\frac{C}{P^u}\right) \\ 0 & , \quad S_0(1-B) \leq S_t < S_0 \\ (S_t - S_0) + S_0 B & , \quad S_t \leq S_0(1-B) \end{cases}$$

(b)

(i) Calculate the risk budget of the spVA for a notional amount of $100.

(ii) Calculate the price/cost of the portfolio of options specified in part a) for the following combinations of participation rates and cap rates (table below)

Price of Option Portfolio	**Participation Rate**		
Cap rate	100%	110%	125%
17%			
26%			
35%			
44%			
53%			
62%			

(iii) Determine the cap rate % (C) of the spVA with participation rate (P^u) = 125% that leads to cost of the portfolio of options in part (b)(ii) that is closest to the risk budget calculated in part (b)(i).

(iv) Explain whether the cap rate increases, decreases, or remains unchanged while keeping the same cost of portfolio of options and increasing the participation rate.

11. Continued

Commentary on Question:
Candidates performed really well on part (b) (i). Part (b) (ii) required the candidates to update the Excel formulas to determine the correct strikes, specify the correct option parameters and update the number/units of long/short position in each option. The data table in Excel would then automatically populate the total option price. Candidates struggled with this question and not many made all of the required updates to get to the final answer. Part (b) (iii) was a straightforward interpretation of the table populated in part (b) (ii) and credit awarded for the correct interpretation. Many candidates were able to recognize the inverse relationship between cap rates and participation rates in part (b) (iv).

All parts answered in Excel.

(c) Derive the portfolio of options and a bond (with maturity value of S_0) that replicate the payoff of this spVA.

Commentary on Question:

Most candidates entirely missed part (c). It required them to algebraically derive the required strike for the additional long put that is needed to support a floored payoff. Limited partial credit was given if candidates determined the correct strike for a specific numerical case but didn't demonstrate how they got to that strike.

The bond maturity payoff is S_0. So, deducting the spVA payoff by S_0 should provide the following maturity payoff equations for the portfolio of options

$$Payoff_t^{spVA} - S_0 = \begin{cases} S_0\mathrm{C} & , \quad S_t \geq S_0\left(1+\dfrac{C}{P^u}\right) \\ P^u(S_t - S_0) & , \quad S_0 \leq S_t < S_0\left(1+\dfrac{C}{P^u}\right) \\ 0 & , \quad S_0(1-B) \leq S_t < S_0 \\ S_t - S_0(1-B) & , S_0(1-B-(1-F)) \leq S_t < S_0(1-B) \\ -S_0(1-F) & , S_t < S_0(1-B-(1-F)) \end{cases}$$

Suppose that the payoff of the additional put option is X when $S_t < S_0(1-B-(1-F))$, then ….

$$S_t - S_0(1-B) + X = -S_0(1-F), when\ S_t < S_0(1-B-(1-F))$$

Thus, X = $S_0(1-B-(1-F)) - S_t\ , when\ S_t < S_0(F-B)$

11. Continued

Therefore,

a) Long P^u units of call option with strike price S_0 (i.e., at-the-money ATM)

b) Short P^u units of call option with strike price $S_0(1 + \frac{C}{P^u})$ (i.e., out-of-money OTM)

c) Short a put option with the strike price $S_0(1 - B)$ (i.e., OTM)

d) Long a put option with the strike price $S_0(F - B)$ (i.e., OTM)