Tables for Exam FAM

The reading material for Exam FAM includes a variety of textbooks. Each text has a set of probability distributions that are used in its readings. For those distributions used in more than one text, the choices of parameterization may not be the same in all of the books. This may be of educational value while you study, but could add a layer of uncertainty in the examination. For this latter reason, we have adopted one set of parameterizations to be used in examinations. This set will be based on Appendices A & B of *Loss Models: From Data to Decisions* by Klugman, Panjer and Willmot. A slightly revised version of these appendices is included in this note. A copy of this note will also be distributed to each candidate at the examination.

Each text also has its own system of dedicated notation and terminology. Sometimes these may conflict. If alternative meanings could apply in an examination question, the symbols will be defined.

When using the normal approximation to a discrete distribution, use the continuity correction.

The density function for the standard normal distribution is $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.

Excerpts from the Appendices to *Loss Models: From Data to Decisions, 5th edition*

September 12, 2018

Appendix A

An Inventory of Continuous Distributions

A.1 Introduction

The incomplete gamma function is given by

$$\Gamma(\alpha; x) = \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t}\, dt, \quad \alpha > 0,\ x > 0,$$

with

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt, \quad \alpha > 0.$$

Also, define

$$G(\alpha; x) = \int_x^\infty t^{\alpha-1} e^{-t}\, dt, \quad x > 0.$$

At times we will need this integral for nonpositive values of α. Integration by parts produces the relationship

$$G(\alpha; x) = -\frac{x^\alpha e^{-x}}{\alpha} + \frac{1}{\alpha} G(\alpha + 1; x).$$

This process can be repeated until the first argument of G is $\alpha + k$, a positive number. Then it can be evaluated from

$$G(\alpha + k; x) = \Gamma(\alpha + k)[1 - \Gamma(\alpha + k; x)].$$

The incomplete beta function is given by

$$\beta(a, b; x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^x t^{a-1}(1-t)^{b-1}\, dt, \quad a > 0,\ b > 0,\ 0 < x < 1.$$

A.2 Transformed Beta Family

A.2.2 Three-Parameter Distributions

A.2.2.1 Generalized Pareto—α, θ, τ

$$
\begin{aligned}
f(x) &= \frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)} \frac{\theta^\alpha x^{\tau-1}}{(x+\theta)^{\alpha+\tau}}, \qquad F(x) = \beta(\tau, \alpha; u), \quad u = \frac{x}{x+\theta}, \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(\tau+k)\Gamma(\alpha-k)}{\Gamma(\alpha)\Gamma(\tau)}, \quad -\tau < k < \alpha, \\
\mathrm{E}[X^k] &= \frac{\theta^k \tau(\tau+1)\cdots(\tau+k-1)}{(\alpha-1)\cdots(\alpha-k)} \quad \text{if } k \text{ is a positive integer}, \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\tau+k)\Gamma(\alpha-k)}{\Gamma(\alpha)\Gamma(\tau)} \beta(\tau+k, \alpha-k; u) + x^k[1-F(x)], \quad k > -\tau, \\
\text{Mode} &= \theta \frac{\tau-1}{\alpha+1}, \quad \tau > 1, \text{ else } 0.
\end{aligned}
$$

A.2.2.2 Burr—α, θ, γ

(Burr Type XII, Singh–Maddala)

$$
\begin{aligned}
f(x) &= \frac{\alpha\gamma(x/\theta)^\gamma}{x[1+(x/\theta)^\gamma]^{\alpha+1}}, \qquad F(x) = 1 - u^\alpha, \quad u = \frac{1}{1+(x/\theta)^\gamma}, \\
\mathrm{VaR}_p(X) &= \theta[(1-p)^{-1/\alpha} - 1]^{1/\gamma}, \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(1+k/\gamma)\Gamma(\alpha-k/\gamma)}{\Gamma(\alpha)}, \quad -\gamma < k < \alpha\gamma, \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(1+k/\gamma)\Gamma(\alpha-k/\gamma)}{\Gamma(\alpha)} \beta(1+k/\gamma, \alpha-k/\gamma; 1-u) + x^k u^\alpha, \quad k > -\gamma, \\
\text{Mode} &= \theta \left(\frac{\gamma-1}{\alpha\gamma+1} \right)^{1/\gamma}, \quad \gamma > 1, \text{ else } 0.
\end{aligned}
$$

A.2.2.3 Inverse Burr—τ, θ, γ

$$
\begin{aligned}
f(x) &= \frac{\tau\gamma(x/\theta)^{\gamma\tau}}{x[1+(x/\theta)^\gamma]^{\tau+1}}, \qquad F(x) = u^\tau, \quad u = \frac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma}, \\
\mathrm{VaR}_p(X) &= \theta(p^{-1/\tau}-1)^{-1/\gamma}, \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(\tau+k/\gamma)\Gamma(1-k/\gamma)}{\Gamma(\tau)}, \quad -\tau\gamma < k < \gamma, \\
\mathrm{E}[(X\wedge x)^k] &= \frac{\theta^k\Gamma(\tau+k/\gamma)\Gamma(1-k/\gamma)}{\Gamma(\tau)}\beta(\tau+k/\gamma, 1-k/\gamma; u) + x^k[1-u^\tau], \quad k > -\tau\gamma, \\
\mathrm{Mode} &= \theta\left(\frac{\tau\gamma-1}{\gamma+1}\right)^{1/\gamma}, \quad \tau\gamma > 1, \text{ else } 0.
\end{aligned}
$$

A.2.3 Two-Parameter Distributions

A.2.3.1 Pareto—α, θ

(Pareto Type II, Lomax)

$$
\begin{aligned}
f(x) &= \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}, \qquad F(x) = 1-\left(\frac{\theta}{x+\theta}\right)^\alpha, \\
\mathrm{VaR}_p(X) &= \theta[(1-p)^{-1/\alpha}-1], \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}, \quad -1 < k < \alpha, \\
\mathrm{E}[X^k] &= \frac{\theta^k k!}{(\alpha-1)\cdots(\alpha-k)}, \quad \text{if } k \text{ is a positive integer}, \\
\mathrm{E}[X\wedge x] &= \frac{\theta}{\alpha-1}\left[1-\left(\frac{\theta}{x+\theta}\right)^{\alpha-1}\right], \quad \alpha \neq 1, \\
\mathrm{E}[X\wedge x] &= -\theta\ln\left(\frac{\theta}{x+\theta}\right), \quad \alpha = 1, \\
\mathrm{TVaR}_p(X) &= \mathrm{VaR}_p(X) + \frac{\theta(1-p)^{-1/\alpha}}{\alpha-1}, \quad \alpha > 1, \\
\mathrm{E}[(X\wedge x)^k] &= \frac{\theta^k\Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}\beta[k+1, \alpha-k; x/(x+\theta)] \\
&\quad + x^k\left(\frac{\theta}{x+\theta}\right)^\alpha, \quad k > -1,\, k \neq \alpha, \\
\mathrm{E}[(X\wedge x)^\alpha] &= \theta^\alpha\left(\frac{x}{x+\theta}\right)^\alpha\left[1+\alpha\sum_{n=0}^\infty \frac{[x/(x+\theta)]^{n+1}}{\alpha+n+1}\right], \\
\mathrm{Mode} &= 0.
\end{aligned}
$$

A.2.3.2 Inverse Pareto—τ, θ

$$\begin{aligned}
f(x) &= \frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}}, \qquad F(x) = \left(\frac{x}{x+\theta}\right)^{\tau}, \\
\mathrm{VaR}_p(X) &= \theta[p^{-1/\tau}-1]^{-1}, \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(\tau+k)\Gamma(1-k)}{\Gamma(\tau)}, \quad -\tau < k < 1, \\
\mathrm{E}[X^k] &= \frac{\theta^k(-k)!}{(\tau-1)\cdots(\tau+k)} \quad \text{if } k \text{ is a negative integer}, \\
\mathrm{E}[(X\wedge x)^k] &= \theta^k\tau\int_0^{x/(x+\theta)} y^{\tau+k-1}(1-y)^{-k}dy + x^k\left[1-\left(\frac{x}{x+\theta}\right)^{\tau}\right], \quad k > -\tau, \\
\text{Mode} &= \theta\frac{\tau-1}{2}, \quad \tau > 1, \text{ else } 0.
\end{aligned}$$

A.2.3.3 Loglogistic—γ, θ

(Fisk)

$$\begin{aligned}
f(x) &= \frac{\gamma(x/\theta)^{\gamma}}{x[1+(x/\theta)^{\gamma}]^2}, \qquad F(x) = u, \quad u = \frac{(x/\theta)^{\gamma}}{1+(x/\theta)^{\gamma}}, \\
\mathrm{VaR}_p(X) &= \theta(p^{-1}-1)^{-1/\gamma}, \\
\mathrm{E}[X^k] &= \theta^k\Gamma(1+k/\gamma)\Gamma(1-k/\gamma), \quad -\gamma < k < \gamma, \\
\mathrm{E}[(X\wedge x)^k] &= \theta^k\Gamma(1+k/\gamma)\Gamma(1-k/\gamma)\beta(1+k/\gamma, 1-k/\gamma; u) + x^k(1-u), \quad k > -\gamma, \\
\text{Mode} &= \theta\left(\frac{\gamma-1}{\gamma+1}\right)^{1/\gamma}, \quad \gamma > 1, \text{ else } 0.
\end{aligned}$$

A.2.3.4 Paralogistic—α, θ

This is a Burr distribution with $\gamma = \alpha$.

$$\begin{aligned}
f(x) &= \frac{\alpha^2(x/\theta)^{\alpha}}{x[1+(x/\theta)^{\alpha}]^{\alpha+1}}, \qquad F(x) = 1-u^{\alpha}, \quad u = \frac{1}{1+(x/\theta)^{\alpha}}, \\
\mathrm{VaR}_p(X) &= \theta[(1-p)^{-1/\alpha}-1]^{1/\alpha}, \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(1+k/\alpha)\Gamma(\alpha-k/\alpha)}{\Gamma(\alpha)}, \quad -\alpha < k < \alpha^2, \\
\mathrm{E}[(X\wedge x)^k] &= \frac{\theta^k\Gamma(1+k/\alpha)\Gamma(\alpha-k/\alpha)}{\Gamma(\alpha)}\beta(1+k/\alpha, \alpha-k/\alpha; 1-u) + x^k u^{\alpha}, \quad k > -\alpha, \\
\text{Mode} &= \theta\left(\frac{\alpha-1}{\alpha^2+1}\right)^{1/\alpha}, \quad \alpha > 1, \text{ else } 0.
\end{aligned}$$

A.2.3.5 Inverse Paralogistic—τ, θ

This is an inverse Burr distribution with $\gamma = \tau$.

$$
\begin{aligned}
f(x) &= \frac{\tau^2 (x/\theta)^{\tau^2}}{x[1+(x/\theta)^\tau]^{\tau+1}}, \qquad F(x) = u^\tau, \quad u = \frac{(x/\theta)^\tau}{1+(x/\theta)^\tau}, \\
\mathrm{VaR}_p(X) &= \theta(p^{-1/\tau} - 1)^{-1/\tau}, \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(\tau + k/\tau)\Gamma(1-k/\tau)}{\Gamma(\tau)}, \quad -\tau^2 < k < \tau, \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\tau + k/\tau)\Gamma(1-k/\tau)}{\Gamma(\tau)} \beta(\tau + k/\tau, 1 - k/\tau; u) + x^k[1-u^\tau], \quad k > -\tau^2, \\
\text{Mode} &= \theta\,(\tau - 1)^{1/\tau}, \quad \tau > 1, \text{ else } 0.
\end{aligned}
$$

A.3 Transformed Gamma Family

A.3.2 Two-Parameter Distributions

A.3.2.1 Gamma—α, θ

(When $\alpha = n/2$ and $\theta = 2$, it is a chi-square distribution with n degrees of freedom.)

$$
\begin{aligned}
f(x) &= \frac{(x/\theta)^\alpha e^{-x/\theta}}{x\Gamma(\alpha)}, \qquad F(x) = \Gamma(\alpha; x/\theta), \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(\alpha + k)}{\Gamma(\alpha)}, \quad k > -\alpha, \\
\mathrm{E}[X^k] &= \theta^k (\alpha + k - 1) \cdots \alpha \quad \text{if } k \text{ is a positive integer}, \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\alpha + k)}{\Gamma(\alpha)} \Gamma(\alpha + k; x/\theta) + x^k[1 - \Gamma(\alpha; x/\theta)], \quad k > -\alpha, \\
\mathrm{E}[(X \wedge x)^k] &= \alpha(\alpha+1)\cdots(\alpha+k-1)\theta^k \Gamma(\alpha+k; x/\theta) \\
&\quad + x^k[1 - \Gamma(\alpha; x/\theta)] \quad \text{if } k \text{ is a positive integer}, \\
M(t) &= (1 - \theta t)^{-\alpha}, \quad t < 1/\theta, \\
\text{Mode} &= \theta(\alpha - 1), \quad \alpha > 1, \text{ else } 0.
\end{aligned}
$$

A.3.2.2 Inverse Gamma—α, θ

(Vinci)

$$
\begin{aligned}
f(x) &= \frac{(\theta/x)^\alpha e^{-\theta/x}}{x\Gamma(\alpha)}, \qquad F(x) = 1 - \Gamma(\alpha; \theta/x), \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(\alpha - k)}{\Gamma(\alpha)}, \quad k < \alpha, \\
\mathrm{E}[X^k] &= \frac{\theta^k}{(\alpha-1)\cdots(\alpha-k)} \quad \text{if } k \text{ is a positive integer,} \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\alpha - k)}{\Gamma(\alpha)}[1 - \Gamma(\alpha - k; \theta/x)] + x^k \Gamma(\alpha; \theta/x) \\
&= \frac{\theta^k G(\alpha - k; \theta/x)}{\Gamma(\alpha)} + x^k \Gamma(\alpha; \theta/x), \text{ all } k, \\
\text{Mode} &= \theta/(\alpha + 1).
\end{aligned}
$$

A.3.2.3 Weibull—θ, τ

$$
\begin{aligned}
f(x) &= \frac{\tau (x/\theta)^\tau e^{-(x/\theta)^\tau}}{x}, \qquad F(x) = 1 - e^{-(x/\theta)^\tau}, \\
\mathrm{VaR}_p(X) &= \theta[-\ln(1-p)]^{1/\tau}, \\
\mathrm{E}[X^k] &= \theta^k \Gamma(1 + k/\tau), \quad k > -\tau, \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k \Gamma(1 + k/\tau) \Gamma[1 + k/\tau; (x/\theta)^\tau] + x^k e^{-(x/\theta)^\tau}, \quad k > -\tau, \\
\text{Mode} &= \theta \left(\frac{\tau - 1}{\tau}\right)^{1/\tau}, \quad \tau > 1, \text{ else } 0.
\end{aligned}
$$

A.3.2.4 Inverse Weibull—θ, τ

(log-Gompertz)

$$
\begin{aligned}
f(x) &= \frac{\tau (\theta/x)^\tau e^{-(\theta/x)^\tau}}{x}, \qquad F(x) = e^{-(\theta/x)^\tau}, \\
\mathrm{VaR}_p(X) &= \theta(-\ln p)^{-1/\tau}, \\
\mathrm{E}[X^k] &= \theta^k \Gamma(1 - k/\tau), \quad k < \tau, \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k \Gamma(1 - k/\tau)\{1 - \Gamma[1 - k/\tau; (\theta/x)^\tau]\} + x^k \left[1 - e^{-(\theta/x)^\tau}\right], \\
&= \theta^k G[1 - k/\tau; (\theta/x)^\tau] + x^k \left[1 - e^{-(\theta/x)^\tau}\right], \quad \text{all } k, \\
\text{Mode} &= \theta \left(\frac{\tau}{\tau + 1}\right)^{1/\tau}.
\end{aligned}
$$

A.3.3 One-Parameter Distributions

A.3.3.1 Exponential—θ

$$
\begin{aligned}
f(x) &= \frac{e^{-x/\theta}}{\theta}, \qquad F(x) = 1 - e^{-x/\theta}, \\
\mathrm{VaR}_p(X) &= -\theta \ln(1-p), \\
\mathrm{E}[X^k] &= \theta^k \Gamma(k+1), \quad k > -1, \\
\mathrm{E}[X^k] &= \theta^k k! \quad \text{if } k \text{ is a positive integer}, \\
\mathrm{E}[X \wedge x] &= \theta(1 - e^{-x/\theta}), \\
\mathrm{TVaR}_p(X) &= -\theta \ln(1-p) + \theta, \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k \Gamma(k+1)\Gamma(k+1; x/\theta) + x^k e^{-x/\theta}, \quad k > -1, \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k k! \Gamma(k+1; x/\theta) + x^k e^{-x/\theta} \quad \text{if } k > -1 \text{ is an integer}, \\
M(z) &= (1 - \theta z)^{-1}, \quad z < 1/\theta, \\
\text{Mode} &= 0.
\end{aligned}
$$

A.3.3.2 Inverse Exponential—θ

$$
\begin{aligned}
f(x) &= \frac{\theta e^{-\theta/x}}{x^2}, \qquad F(x) = e^{-\theta/x}, \\
\mathrm{VaR}_p(X) &= \theta(-\ln p)^{-1}, \\
\mathrm{E}[X^k] &= \theta^k \Gamma(1-k), \quad k < 1, \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k G(1-k; \theta/x) + x^k(1 - e^{-\theta/x}), \quad \text{all } k, \\
\text{Mode} &= \theta/2.
\end{aligned}
$$

A.5 Other Distributions

A.5.1.1 Lognormal—μ,σ

(μ can be negative)

$$
\begin{aligned}
f(x) &= \frac{1}{x\sigma\sqrt{2\pi}} \exp(-z^2/2) = \phi(z)/(\sigma x), \quad z = \frac{\ln x - \mu}{\sigma}, \\
F(x) &= \Phi(z), \\
\mathrm{E}[X^k] &= \exp\left(k\mu + \tfrac{1}{2}k^2\sigma^2\right), \\
\mathrm{E}[(X \wedge x)^k] &= \exp\left(k\mu + \tfrac{1}{2}k^2\sigma^2\right) \Phi\left(\frac{\ln x - \mu - k\sigma^2}{\sigma}\right) + x^k[1 - F(x)], \\
\text{Mode} &= \exp(\mu - \sigma^2).
\end{aligned}
$$

A.5.1.2 Inverse Gaussian—μ, θ

$$
\begin{aligned}
f(x) &= \left(\frac{\theta}{2\pi x^3}\right)^{1/2} \exp\left(-\frac{\theta z^2}{2x}\right), \quad z = \frac{x-\mu}{\mu}, \\
F(x) &= \Phi\left[z\left(\frac{\theta}{x}\right)^{1/2}\right] + \exp\left(\frac{2\theta}{\mu}\right)\Phi\left[-y\left(\frac{\theta}{x}\right)^{1/2}\right], \quad y = \frac{x+\mu}{\mu}, \\
\mathrm{E}[X] &= \mu, \quad \mathrm{Var}[X] = \mu^3/\theta, \\
\mathrm{E}[X^k] &= \sum_{n=0}^{k-1} \frac{(k+n-1)!}{(k-n-1)!n!} \frac{\mu^{n+k}}{(2\theta)^n}, \quad k = 1, 2, \ldots, \\
\mathrm{E}[X \wedge x] &= x - \mu z \Phi\left[z\left(\frac{\theta}{x}\right)^{1/2}\right] - \mu y \exp(2\theta/\mu)\Phi\left[-y\left(\frac{\theta}{x}\right)^{1/2}\right], \\
M(z) &= \exp\left[\frac{\theta}{\mu}\left(1 - \sqrt{1 - \frac{2\mu^2}{\theta}z}\right)\right], \quad z < \frac{\theta}{2\mu^2}.
\end{aligned}
$$

A.5.1.3 Log-t—r, μ, σ

(μ can be negative) Let Y have a t distribution with r degrees of freedom. Then $X = \exp(\sigma Y + \mu)$ has the log-t distribution. Positive moments do not exist for this distribution. Just as the t distribution has a heavier tail than the normal distribution, this distribution has a heavier tail than the lognormal distribution.

$$
\begin{aligned}
f(x) &= \frac{\Gamma\left(\frac{r+1}{2}\right)}{x\sigma\sqrt{\pi r}\,\Gamma\left(\frac{r}{2}\right)\left[1 + \frac{1}{r}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right]^{(r+1)/2}}, \\
F(x) &= F_r\left(\frac{\ln x - \mu}{\sigma}\right) \text{ with } F_r(t) \text{ the cdf of a } t \text{ distribution with } r \text{ df}, \\
F(x) &= \begin{cases}
\frac{1}{2}\beta\left[\frac{r}{2}, \frac{1}{2}; \frac{r}{r + \left(\frac{\ln x - \mu}{\sigma}\right)^2}\right], & 0 < x \le e^\mu, \\
1 - \frac{1}{2}\beta\left[\frac{r}{2}, \frac{1}{2}; \frac{r}{r + \left(\frac{\ln x - \mu}{\sigma}\right)^2}\right], & x \ge e^\mu.
\end{cases}
\end{aligned}
$$

A.5.1.4 Single-Parameter Pareto—α, θ

$$
\begin{aligned}
f(x) &= \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, \quad x > \theta, \qquad F(x) = 1 - \left(\frac{\theta}{x}\right)^\alpha, \quad x > \theta, \\
\mathrm{VaR}_p(X) &= \theta(1-p)^{-1/\alpha}, \\
\mathrm{E}[X^k] &= \frac{\alpha\theta^k}{\alpha - k}, \quad k < \alpha, \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\alpha\theta^k}{\alpha - k} - \frac{k\theta^\alpha}{(\alpha - k)x^{\alpha-k}}, \ x \geq \theta, \ k \neq \alpha, \\
\mathrm{E}[(X \wedge x)^\alpha] &= \theta^\alpha[1 + \alpha \ln(x/\theta)], \\
\mathrm{TVaR}_p(X) &= \frac{\alpha\theta(1-p)^{-1/\alpha}}{\alpha - 1}, \quad \alpha > 1, \\
\text{Mode} &= \theta.
\end{aligned}
$$

Note: Although there appear to be two parameters, only α is a true parameter. The value of θ must be set in advance.

A.6 Distributions with Finite Support

For these two distributions, the scale parameter θ is assumed known.

A.6.1.1 Generalized Beta—a, b, θ, τ

$$
\begin{aligned}
f(x) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^a (1-u)^{b-1} \frac{\tau}{x}, \quad 0 < x < \theta, \quad u = (x/\theta)^\tau, \\
F(x) &= \beta(a, b; u), \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(a+b)\Gamma(a+k/\tau)}{\Gamma(a)\Gamma(a+b+k/\tau)}, \quad k > -a\tau, \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(a+b)\Gamma(a+k/\tau)}{\Gamma(a)\Gamma(a+b+k/\tau)} \beta(a + k/\tau, b; u) + x^k[1 - \beta(a, b; u)].
\end{aligned}
$$

A.6.1.2 Beta—a, b, θ

The case $\theta = 1$ has no special name but is the commonly used version of this distribution.

$$
\begin{aligned}
f(x) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^a (1-u)^{b-1} \frac{1}{x}, \quad 0 < x < \theta, \quad u = x/\theta, \\
F(x) &= \beta(a, b; u), \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(a+b)\Gamma(a+k)}{\Gamma(a)\Gamma(a+b+k)}, \quad k > -a, \\
\mathrm{E}[X^k] &= \frac{\theta^k a(a+1)\cdots(a+k-1)}{(a+b)(a+b+1)\cdots(a+b+k-1)} \quad \text{if } k \text{ is a positive integer}, \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k a(a+1)\cdots(a+k-1)}{(a+b)(a+b+1)\cdots(a+b+k-1)} \beta(a+k, b; u) \\
&\quad + x^k [1 - \beta(a, b; u)].
\end{aligned}
$$

Appendix B

An Inventory of Discrete Distributions

B.1 Introduction

The 16 models presented in this appendix fall into three classes. The divisions are based on the **algorithm** used to compute the probabilities. For some of the more familiar distributions these formulas will look different from the ones you may have learned, but they produce the same probabilities. After each name, the parameters are given. All parameters are positive unless otherwise indicated. In all cases, p_k is the probability of observing k losses.

For finding moments, the most convenient form is to give the factorial moments. The jth factorial moment is $\mu_{(j)} = \mathrm{E}[N(N-1)\cdots(N-j+1)]$. We have $\mathrm{E}[N] = \mu_{(1)}$ and $\mathrm{Var}(N) = \mu_{(2)} + \mu_{(1)} - \mu_{(1)}^2$.

The estimators presented are not intended to be useful estimators but, rather, provide starting values for maximizing the likelihood (or other) function. For determining starting values, the following quantities are used (where n_k is the observed frequency at k [if, for the last entry, n_k represents the number of observations at k or more, assume it was at exactly k] and n is the sample size):

$$\hat{\mu} = \frac{1}{n}\sum_{k=1}^{\infty} k n_k, \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{k=1}^{\infty} k^2 n_k - \hat{\mu}^2.$$

When the method of moments is used to determine the starting value, a circumflex (e.g., $\hat{\lambda}$) is used. For any other method, a tilde (e.g., $\tilde{\lambda}$) is used. When the starting value formulas do not provide admissible parameter values, a truly crude guess is to set the product of all λ and β parameters equal to the sample mean and set all other parameters equal to 1. If there are two λ or β parameters, an easy choice is to set each to the square root of the sample mean.

The last item presented is the probability generating function,

$$P(z) = \mathrm{E}[z^N].$$

B.2 The $(a,b,0)$ Class

The distributions in this class have support on $0,1,\ldots$. For this class, a particular distribution is specified by setting p_0 and then using $p_k=(a+b/k)p_{k-1}$. Specific members are created by setting p_0, a, and b. For any member, $\mu_{(1)}=(a+b)/(1-a)$, and for higher j, $\mu_{(j)}=(aj+b)\mu_{(j-1)}/(1-a)$. The variance is $(a+b)/(1-a)^2$.

B.2.1.1 Poisson—λ

$$
\begin{aligned}
p_0 &= e^{-\lambda}, \quad a=0, \quad b=\lambda, \qquad p_k=\frac{e^{-\lambda}\lambda^k}{k!},\\
\mathrm{E}[N] &= \lambda, \quad \mathrm{Var}[N]=\lambda,\\
\hat{\lambda} &= \hat{\mu},\\
P(z) &= e^{\lambda(z-1)}.
\end{aligned}
$$

B.2.1.2 Geometric—β

$$
\begin{aligned}
p_0 &= \frac{1}{1+\beta}, \quad a=\frac{\beta}{1+\beta}, \quad b=0, \qquad p_k=\frac{\beta^k}{(1+\beta)^{k+1}},\\
\mathrm{E}[N] &= \beta, \quad \mathrm{Var}[N]=\beta(1+\beta),\\
\hat{\beta} &= \hat{\mu},\\
P(z) &= [1-\beta(z-1)]^{-1}, \quad -(1+1/\beta)<z<1+1/\beta.
\end{aligned}
$$

This is a special case of the negative binomial with $r=1$.

B.2.1.3 Binomial—q,m

($0<q<1$, m an integer)

$$
\begin{aligned}
p_0 &= (1-q)^m, \quad a=-\frac{q}{1-q}, \quad b=\frac{(m+1)q}{1-q},\\
p_k &= \binom{m}{k}q^k(1-q)^{m-k}, \quad k=0,1,\ldots,m,\\
\mathrm{E}[N] &= mq, \quad \mathrm{Var}[N]=mq(1-q),\\
\hat{q} &= \hat{\mu}/m,\\
P(z) &= [1+q(z-1)]^m.
\end{aligned}
$$

B.2.1.4 Negative Binomial—β, r

$$\begin{aligned}
p_0 &= (1+\beta)^{-r}, \quad a = \frac{\beta}{1+\beta}, \quad b = \frac{(r-1)\beta}{1+\beta}, \\
p_k &= \frac{r(r+1)\cdots(r+k-1)\beta^k}{k!(1+\beta)^{r+k}}, \\
\mathrm{E}[N] &= r\beta, \quad \mathrm{Var}[N] = r\beta(1+\beta), \\
\hat{\beta} &= \frac{\hat{\sigma}^2}{\hat{\mu}} - 1, \quad \hat{r} = \frac{\hat{\mu}^2}{\hat{\sigma}^2 - \hat{\mu}}, \\
P(z) &= [1-\beta(z-1)]^{-r}, \quad -(1+1/\beta) < z < 1+1/\beta.
\end{aligned}$$

B.3 The $(a, b, 1)$ Class

To distinguish this class from the $(a, b, 0)$ class, the probabilities are denoted $\Pr(N = k) = p_k^M$ or $\Pr(N = k) = p_k^T$ depending on which subclass is being represented. For this class, p_0^M is arbitrary (i.e., it is a parameter), and then p_1^M or p_1^T is a specified function of the parameters a and b. Subsequent probabilities are obtained recursively as in the $(a, b, 0)$ class: $p_k^M = (a + b/k)p_{k-1}^M$, $k = 2, 3, \ldots$, with the same recursion for p_k^T. There are two subclasses of this class. When discussing their members, we often refer to the "corresponding" member of the $(a, b, 0)$ class. This refers to the member of that class with the same values for a and b. The notation p_k will continue to be used for probabilities for the corresponding $(a, b, 0)$ distribution.

B.3.1 The Zero-Truncated Subclass

The members of this class have $p_0^T = 0$, and therefore it need not be estimated. These distributions should only be used when a value of zero is impossible. The first factorial moment is $\mu_{(1)} = (a+b)/[(1-a)(1-p_0)]$, where p_0 is the value for the corresponding member of the $(a, b, 0)$ class. For the logarithmic distribution (which has no corresponding member), $\mu_{(1)} = \beta/\ln(1+\beta)$. Higher factorial moments are obtained recursively with the same formula as with the $(a, b, 0)$ class. The variance is $(a+b)[1-(a+b+1)p_0]/[(1-a)(1-p_0)]^2$. For those members of the subclass that have corresponding $(a, b, 0)$ distributions, $p_k^T = p_k/(1-p_0)$.

B.3.1.1 Zero-Truncated Poisson—λ

$$
\begin{aligned}
p_1^T &= \frac{\lambda}{e^\lambda - 1}, \quad a = 0, \quad b = \lambda, \\
p_k^T &= \frac{\lambda^k}{k!(e^\lambda - 1)}, \\
\mathrm{E}[N] &= \lambda/(1 - e^{-\lambda}), \quad \mathrm{Var}[N] = \lambda[1 - (\lambda + 1)e^{-\lambda}]/(1 - e^{-\lambda})^2, \\
\tilde{\lambda} &= \ln(n\hat{\mu}/n_1), \\
P(z) &= \frac{e^{\lambda z} - 1}{e^\lambda - 1}.
\end{aligned}
$$

B.3.1.2 Zero-Truncated Geometric—β

$$
\begin{aligned}
p_1^T &= \frac{1}{1+\beta}, \quad a = \frac{\beta}{1+\beta}, \quad b = 0, \\
p_k^T &= \frac{\beta^{k-1}}{(1+\beta)^k}, \\
\mathrm{E}[N] &= 1 + \beta, \quad \mathrm{Var}[N] = \beta(1+\beta), \\
\hat{\beta} &= \hat{\mu} - 1, \\
P(z) &= \frac{[1 - \beta(z-1)]^{-1} - (1+\beta)^{-1}}{1 - (1+\beta)^{-1}}, \quad -(1 + 1/\beta) < z < 1 + 1/\beta.
\end{aligned}
$$

This is a special case of the zero-truncated negative binomial with $r = 1$.

B.3.1.3 Logarithmic—β

$$
\begin{aligned}
p_1^T &= \frac{\beta}{(1+\beta)\ln(1+\beta)}, \quad a = \frac{\beta}{1+\beta}, \quad b = -\frac{\beta}{1+\beta}, \\
p_k^T &= \frac{\beta^k}{k(1+\beta)^k \ln(1+\beta)}, \\
\mathrm{E}[N] &= \beta/\ln(1+\beta), \quad \mathrm{Var}[N] = \frac{\beta[1 + \beta - \beta/\ln(1+\beta)]}{\ln(1+\beta)}, \\
\tilde{\beta} &= \frac{n\hat{\mu}}{n_1} - 1 \quad \text{or} \quad \frac{2(\hat{\mu} - 1)}{\hat{\mu}}, \\
P(z) &= 1 - \frac{\ln[1 - \beta(z-1)]}{\ln(1+\beta)}, \quad -(1 + 1/\beta) < z < 1 + 1/\beta.
\end{aligned}
$$

This is a limiting case of the zero-truncated negative binomial as $r \to 0$.

B.3.1.4 Zero-Truncated Binomial—$q, m,$

$(0 < q < 1,\ m$ an integer)

$$
\begin{aligned}
p_1^T &= \frac{m(1-q)^{m-1}q}{1-(1-q)^m}, \quad a = -\frac{q}{1-q}, \quad b = \frac{(m+1)q}{1-q},\\
p_k^T &= \frac{\binom{m}{k}q^k(1-q)^{m-k}}{1-(1-q)^m}, \quad k = 1, 2, \ldots, m,\\
\mathrm{E}[N] &= \frac{mq}{1-(1-q)^m},\\
\mathrm{Var}[N] &= \frac{mq[(1-q)-(1-q+mq)(1-q)^m]}{[1-(1-q)^m]^2},\\
\tilde{q} &= \frac{\hat{\mu}}{m},\\
P(z) &= \frac{[1+q(z-1)]^m-(1-q)^m}{1-(1-q)^m},.
\end{aligned}
$$

B.3.1.5 Zero-Truncated Negative Binomial—$\beta, r\ (r > -1, r \neq 0)$

$$
\begin{aligned}
p_1^T &= \frac{r\beta}{(1+\beta)^{r+1}-(1+\beta)}, \quad a = \frac{\beta}{1+\beta}, \quad b = \frac{(r-1)\beta}{1+\beta},\\
p_k^T &= \frac{r(r+1)\cdots(r+k-1)}{k![(1+\beta)^r-1]}\left(\frac{\beta}{1+\beta}\right)^k,\\
\mathrm{E}[N] &= \frac{r\beta}{1-(1+\beta)^{-r}},\\
\mathrm{Var}[N] &= \frac{r\beta[(1+\beta)-(1+\beta+r\beta)(1+\beta)^{-r}]}{[1-(1+\beta)^{-r}]^2},\\
\tilde{\beta} &= \frac{\hat{\sigma}^2}{\hat{\mu}}-1, \quad \tilde{r} = \frac{\hat{\mu}^2}{\hat{\sigma}^2-\hat{\mu}},\\
P(z) &= \frac{[1-\beta(z-1)]^{-r}-(1+\beta)^{-r}}{1-(1+\beta)^{-r}}, \quad -(1+1/\beta) < z < 1+1/\beta.
\end{aligned}
$$

This distribution is sometimes called the extended truncated negative binomial distribution because the parameter r can extend below zero.

B.3.2 The Zero-Modified Subclass

A zero-modified distribution is created by starting with a truncated distribution and then placing an arbitrary amount of probability at zero. This probability, p_0^M, is a parameter. The remaining probabilities are adjusted accordingly. Values of p_k^M can be determined from the corresponding zero-truncated distribution as $p_k^M = (1 - p_0^M)p_k^T$ or from the corresponding $(a, b, 0)$ distribution as $p_k^M = (1 - p_0^M)p_k/(1 - p_0)$. The same recursion used for the zero-truncated subclass applies.

The mean is $1 - p_0^M$ times the mean for the corresponding zero-truncated distribution. The variance is $1 - p_0^M$ times the zero-truncated variance plus $p_0^M(1 - p_0^M)$ times the square of the zero-truncated mean. The probability generating function is $P^M(z) = p_0^M + (1 - p_0^M)P(z)$, where $P(z)$ is the probability generating function for the corresponding zero-truncated distribution.

The maximum likelihood estimator of p_0^M is always the sample relative frequency at zero.

Standard Ultimate Life Table: Basic Functions and Single Net Premiums at i = 0.05

x	l_x	q_x	$\ddot{a}_x$	A_x	${}^2A_x$	$\ddot{a}_{x:\overline{10}\vert}$	$A_{x:\overline{10}\vert}$	$\ddot{a}_{x:\overline{20}\vert}$	$A_{x:\overline{20}\vert}$	${}_5E_x$	${}_{10}E_x$	${}_{20}E_x$	x
20	100,000.0	0.000250	19.9664	0.04922	0.00580	8.0991	0.61433	13.0559	0.37829	0.78252	0.61224	0.37440	20
21	99,975.0	0.000253	19.9197	0.05144	0.00614	8.0990	0.61433	13.0551	0.37833	0.78250	0.61220	0.37429	21
22	99,949.7	0.000257	19.8707	0.05378	0.00652	8.0988	0.61434	13.0541	0.37837	0.78248	0.61215	0.37417	22
23	99,924.0	0.000262	19.8193	0.05622	0.00694	8.0986	0.61435	13.0531	0.37842	0.78245	0.61210	0.37404	23
24	99,897.8	0.000267	19.7655	0.05879	0.00739	8.0983	0.61437	13.0519	0.37848	0.78243	0.61205	0.37390	24
25	99,871.1	0.000273	19.7090	0.06147	0.00788	8.0981	0.61438	13.0506	0.37854	0.78240	0.61198	0.37373	25
26	99,843.8	0.000280	19.6499	0.06429	0.00841	8.0978	0.61439	13.0491	0.37862	0.78236	0.61191	0.37354	26
27	99,815.9	0.000287	19.5878	0.06725	0.00900	8.0974	0.61441	13.0474	0.37869	0.78233	0.61183	0.37334	27
28	99,787.2	0.000296	19.5228	0.07034	0.00964	8.0970	0.61443	13.0455	0.37878	0.78229	0.61174	0.37310	28
29	99,757.7	0.000305	19.4547	0.07359	0.01033	8.0966	0.61445	13.0434	0.37888	0.78224	0.61163	0.37284	29
30	99,727.3	0.000315	19.3834	0.07698	0.01109	8.0961	0.61447	13.0410	0.37900	0.78219	0.61152	0.37254	30
31	99,695.8	0.000327	19.3086	0.08054	0.01192	8.0956	0.61450	13.0384	0.37913	0.78213	0.61139	0.37221	31
32	99,663.2	0.000341	19.2303	0.08427	0.01281	8.0949	0.61453	13.0354	0.37927	0.78206	0.61124	0.37183	32
33	99,629.3	0.000356	19.1484	0.08817	0.01379	8.0943	0.61456	13.0320	0.37943	0.78199	0.61108	0.37141	33
34	99,593.8	0.000372	19.0626	0.09226	0.01486	8.0935	0.61460	13.0282	0.37961	0.78190	0.61090	0.37094	34
35	99,556.7	0.000391	18.9728	0.09653	0.01601	8.0926	0.61464	13.0240	0.37981	0.78181	0.61069	0.37041	35
36	99,517.8	0.000412	18.8788	0.10101	0.01727	8.0916	0.61468	13.0192	0.38004	0.78170	0.61046	0.36982	36
37	99,476.7	0.000436	18.7805	0.10569	0.01863	8.0905	0.61474	13.0138	0.38029	0.78158	0.61020	0.36915	37
38	99,433.3	0.000463	18.6777	0.11059	0.02012	8.0893	0.61480	13.0078	0.38058	0.78145	0.60990	0.36841	38
39	99,387.3	0.000493	18.5701	0.11571	0.02173	8.0879	0.61486	13.0011	0.38090	0.78130	0.60957	0.36757	39
40	99,338.3	0.000527	18.4578	0.12106	0.02347	8.0863	0.61494	12.9935	0.38126	0.78113	0.60920	0.36663	40
41	99,285.9	0.000565	18.3403	0.12665	0.02536	8.0846	0.61502	12.9850	0.38167	0.78094	0.60879	0.36558	41
42	99,229.8	0.000608	18.2176	0.13249	0.02741	8.0826	0.61511	12.9754	0.38212	0.78072	0.60832	0.36440	42
43	99,169.4	0.000656	18.0895	0.13859	0.02963	8.0804	0.61522	12.9647	0.38263	0.78048	0.60780	0.36307	43
44	99,104.3	0.000710	17.9558	0.14496	0.03203	8.0779	0.61534	12.9526	0.38321	0.78021	0.60721	0.36159	44
45	99,033.9	0.000771	17.8162	0.15161	0.03463	8.0751	0.61547	12.9391	0.38385	0.77991	0.60655	0.35994	45
46	98,957.6	0.000839	17.6706	0.15854	0.03744	8.0720	0.61562	12.9240	0.38457	0.77956	0.60581	0.35809	46
47	98,874.5	0.000916	17.5189	0.16577	0.04047	8.0684	0.61579	12.9070	0.38538	0.77918	0.60498	0.35601	47
48	98,783.9	0.001003	17.3607	0.17330	0.04374	8.0645	0.61598	12.8880	0.38629	0.77875	0.60404	0.35370	48
49	98,684.9	0.001100	17.1960	0.18114	0.04727	8.0600	0.61619	12.8667	0.38730	0.77827	0.60299	0.35112	49
50	98,576.4	0.001209	17.0245	0.18931	0.05108	8.0550	0.61643	12.8428	0.38844	0.77772	0.60182	0.34824	50
51	98,457.2	0.001331	16.8461	0.19780	0.05517	8.0494	0.61670	12.8161	0.38971	0.77711	0.60050	0.34503	51
52	98,326.2	0.001469	16.6606	0.20664	0.05957	8.0431	0.61700	12.7862	0.39113	0.77643	0.59902	0.34146	52
53	98,181.8	0.001623	16.4678	0.21582	0.06430	8.0360	0.61733	12.7527	0.39273	0.77566	0.59736	0.33749	53
54	98,022.4	0.001797	16.2676	0.22535	0.06938	8.0281	0.61771	12.7154	0.39451	0.77479	0.59550	0.33308	54
55	97,846.2	0.001993	16.0599	0.23524	0.07483	8.0192	0.61813	12.6737	0.39649	0.77382	0.59342	0.32819	55
56	97,651.2	0.002212	15.8444	0.24550	0.08067	8.0092	0.61861	12.6271	0.39871	0.77273	0.59109	0.32279	56
57	97,435.2	0.002459	15.6212	0.25613	0.08692	7.9980	0.61914	12.5752	0.40118	0.77151	0.58848	0.31681	57
58	97,195.6	0.002736	15.3901	0.26714	0.09360	7.9854	0.61974	12.5174	0.40393	0.77014	0.58556	0.31024	58
59	96,929.6	0.003048	15.1511	0.27852	0.10073	7.9713	0.62041	12.4531	0.40700	0.76860	0.58229	0.30300	59
60	96,634.1	0.003398	14.9041	0.29028	0.10834	7.9555	0.62116	12.3816	0.41040	0.76687	0.57864	0.29508	60

Standard Ultimate Life Table: Basic Functions and Single Net Premiums at i = 0.05

x	l_x	q_x	$\ddot{a}_x$	A_x	$^{2}A_x$	$\ddot{a}_{x:\overline{10}\rvert}$	$A_{x:\overline{10}\rvert}$	$\ddot{a}_{x:\overline{20}\rvert}$	$A_{x:\overline{20}\rvert}$	$_{5}E_x$	$_{10}E_x$	$_{20}E_x$	x
61	96,305.8	0.003792	14.6491	0.30243	0.11644	7.9379	0.62201	12.3024	0.41417	0.76493	0.57457	0.28641	61
62	95,940.6	0.004234	14.3861	0.31495	0.12506	7.9181	0.62295	12.2145	0.41836	0.76276	0.57003	0.27698	62
63	95,534.4	0.004730	14.1151	0.32785	0.13421	7.8960	0.62400	12.1174	0.42298	0.76033	0.56496	0.26674	63
64	95,082.5	0.005288	13.8363	0.34113	0.14392	7.8712	0.62518	12.0101	0.42809	0.75760	0.55932	0.25569	64
65	94,579.7	0.005915	13.5498	0.35477	0.15420	7.8435	0.62650	11.8920	0.43371	0.75455	0.55305	0.24381	65
66	94,020.3	0.006619	13.2557	0.36878	0.16507	7.8126	0.62797	11.7622	0.43990	0.75114	0.54609	0.23112	66
67	93,398.1	0.007409	12.9542	0.38313	0.17654	7.7781	0.62961	11.6199	0.44667	0.74732	0.53836	0.21764	67
68	92,706.1	0.008297	12.6456	0.39783	0.18862	7.7396	0.63145	11.4643	0.45408	0.74305	0.52981	0.20343	68
69	91,936.9	0.009294	12.3302	0.41285	0.20133	7.6968	0.63349	11.2949	0.46215	0.73828	0.52036	0.18856	69
70	91,082.4	0.010413	12.0083	0.42818	0.21467	7.6491	0.63576	11.1109	0.47091	0.73295	0.50994	0.17313	70
71	90,134.0	0.011670	11.6803	0.44379	0.22864	7.5961	0.63828	10.9118	0.48039	0.72701	0.49848	0.15730	71
72	89,082.1	0.013081	11.3468	0.45968	0.24324	7.5373	0.64108	10.6974	0.49060	0.72039	0.48590	0.14122	72
73	87,916.8	0.014664	11.0081	0.47580	0.25847	7.4721	0.64419	10.4675	0.50155	0.71303	0.47215	0.12511	73
74	86,627.6	0.016440	10.6649	0.49215	0.27433	7.3999	0.64762	10.2221	0.51323	0.70483	0.45715	0.10918	74
75	85,203.5	0.018433	10.3178	0.50868	0.29079	7.3203	0.65142	9.9616	0.52564	0.69574	0.44085	0.09368	75
76	83,632.9	0.020668	9.9674	0.52536	0.30783	7.2325	0.65560	9.6866	0.53873	0.68566	0.42323	0.07887	76
77	81,904.3	0.023175	9.6145	0.54217	0.32544	7.1360	0.66019	9.3980	0.55247	0.67450	0.40427	0.06500	77
78	80,006.2	0.025984	9.2598	0.55906	0.34359	7.0302	0.66523	9.0970	0.56681	0.66217	0.38396	0.05230	78
79	77,927.4	0.029132	8.9042	0.57599	0.36224	6.9146	0.67074	8.7850	0.58166	0.64859	0.36235	0.04096	79
80	75,657.2	0.032658	8.5484	0.59293	0.38134	6.7885	0.67674	8.4639	0.59696	0.63365	0.33952	0.03113	80
81	73,186.3	0.036607	8.1934	0.60984	0.40086	6.6517	0.68325	8.1354	0.61260	0.61727	0.31556	0.02286	81
82	70,507.2	0.041025	7.8401	0.62666	0.42075	6.5037	0.69030	7.8018	0.62848	0.59936	0.29064	0.01616	82
83	67,614.6	0.045968	7.4893	0.64336	0.44094	6.3443	0.69789	7.4651	0.64452	0.57985	0.26498	0.01094	83
84	64,506.5	0.051493	7.1421	0.65990	0.46137	6.1735	0.70602	7.1275	0.66059	0.55868	0.23882	0.00706	84
85	61,184.9	0.057665	6.7993	0.67622	0.48199	5.9915	0.71469	6.7910	0.67662	0.53581	0.21250	0.00431	85
86	57,656.7	0.064554	6.4619	0.69229	0.50272	5.7986	0.72388	6.4574	0.69250	0.51122	0.18635	0.00248	86
87	53,934.7	0.072237	6.1308	0.70806	0.52349	5.5954	0.73355	6.1285	0.70817	0.48492	0.16079	0.00133	87
88	50,038.6	0.080798	5.8068	0.72349	0.54422	5.3828	0.74368	5.8057	0.72354	0.45697	0.13621	0.00066	88
89	45,995.6	0.090326	5.4908	0.73853	0.56484	5.1620	0.75419	5.4903	0.73856	0.42748	0.11305	0.00030	89
90	41,841.1	0.100917	5.1835	0.75317	0.58528	4.9346	0.76502	5.1833	0.75317	0.39659	0.09168	0.00012	90
91	37,618.6	0.112675	4.8858	0.76735	0.60545	4.7021	0.77609	4.8857	0.76735	0.36453	0.07244	0.00005	91
92	33,379.9	0.125708	4.5981	0.78104	0.62529	4.4665	0.78731	4.5981	0.78104	0.33158	0.05559	0.00002	92
93	29,183.8	0.140128	4.3213	0.79423	0.64472	4.2299	0.79858	4.3213	0.79423	0.29808	0.04128	0.00000	93
94	25,094.3	0.156052	4.0556	0.80688	0.66368	3.9945	0.80979	4.0556	0.80688	0.26445	0.02955	0.00000	94
95	21,178.3	0.173599	3.8017	0.81897	0.68209	3.7624	0.82084	3.8017	0.81897	0.23116	0.02029	0.00000	95
96	17,501.8	0.192887	3.5597	0.83049	0.69991	3.5356	0.83164	3.5597	0.83049	0.19872	0.01330	0.00000	96
97	14,125.9	0.214030	3.3300	0.84143	0.71708	3.3159	0.84210	3.3300	0.84143	0.16765	0.00827	0.00000	97
98	11,102.5	0.237134	3.1127	0.85177	0.73356	3.1050	0.85214	3.1127	0.85177	0.13850	0.00485	0.00000	98
99	8,469.7	0.262294	2.9079	0.86153	0.74930	2.9039	0.86172	2.9079	0.86153	0.11173	0.00266	0.00000	99
100	6,248.2	0.289584	2.7156	0.87068	0.76427	2.7137	0.87078	2.7156	0.87068	0.08777	0.00136	0.00000	100

Interest Functions

Interest Functions at *i* = 0.05

m	$i^{(m)}$	$d^{(m)}$	$i/i^{(m)}$	$d/d^{(m)}$	$\alpha(m)$	$\beta(m)$
1	0.05000	0.04762	1.00000	1.00000	1.00000	0.00000
2	0.04939	0.04820	1.01235	0.98795	1.00015	0.25617
4	0.04909	0.04849	1.01856	0.98196	1.00019	0.38272
12	0.04889	0.04869	1.02271	0.97798	1.00020	0.46651
∞	0.04879	0.04879	1.02480	0.97600	1.00020	0.50823

Long Term Actuarial Mathematics Formulas

Interest Functions

$$\alpha(m) = \frac{id}{i^{(m)}d^{(m)}} \quad \text{and} \quad \beta(m) = \frac{i - i^{(m)}}{i^{(m)}d^{(m)}}$$

Under Makeham's Law

$$\mu_x = A + Bc^x \quad \text{and} \quad {}_t p_x = \exp\left[-At - \frac{B}{\ln c} c^x (c^t - 1)\right]$$

Three-Term Woolhouse Formula in a Single Decrement Context

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x)$$

Greenwood's Approximation Formula

$$V\left[\hat{S}(t)\right] \approx [\hat{S}(t)]^2 \sum_{j:t_{(j)} \le t} \frac{d_j}{r_j(r_j - d_j)} \quad \text{for } t < t_{(\max)}$$

Estimated Variance Formula for Nelson-Aalen Estimator

$$V[\hat{S}(t)] \approx [\hat{S}(t)]^2 \sum_{j:t_{(j)} \le t} \frac{d_j(r_j - d_j)}{r_j^3} \quad \text{for } t < t_{(\max)}$$